

CSIR-UGC-NET/JRF

Previous Year's

SOLVED PAPERS

June 2011 - June 2014

PHYSICAL SCIENCES

First Edition : November 2014

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CSIR-UGC-NET/JRF- JUNE - 2011
PHYSICAL SCIENCES BOOKLET - [A]

PART - B

21. A particle of unit mass moves in a potential $V(x) = ax^2 + \frac{b}{x^2}$, where a and b are positive constants. The angular frequency of small oscillations about the minimum of the potential is:
- (a) $\sqrt{8b}$ (b) $\sqrt{8a}$ (c) $\sqrt{8a/b}$ (d) $\sqrt{8b/a}$

Soln. $V(x) = ax^2 + \frac{b}{x^2}$

Angular frequency, $\omega = \sqrt{\frac{k}{m}}$, where, $k = \left. \frac{d^2V}{dx^2} \right|_{x=x_0} = 2a + \frac{b}{x_0^4}$

Where, x_0 is the point at which $V(x)$ is minimum.

For minimum, $\left. \frac{dV}{dx} \right|_{x=x_0} = 0$

$\Rightarrow 2ax_0 - \frac{2b}{x_0^3} = 0 \Rightarrow x_0^2 = \sqrt{\frac{b}{a}}$

$\therefore k = 2a + \frac{6b}{x_0^4} = 8a$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8a}{1}} = \sqrt{8a}$

Correct answer : (b)

22. A signal of frequency 10 kHz is being digitized by an A/D converter. A possible sampling time which can be used is:
- (a) 100 μs (b) 40 μs (c) 60 μs (d) 200 μs

Soln. Signal frequency = 10 kHz = 10^4 Hz

According to sampling theorem

Minimum frequency for sampling = 2(signal frequency) = 2×10^4 Hz

Sampling time = $\frac{1}{2 \times 10^4} s = \frac{100}{2} \times 10^{-6} s$

Minimum sampling time = 50 μs which is closest to 60 μs .

Correct answer : (c)

23. The electrostatic potential $V(x, y)$ in free space in a region where the charge density ρ is zero is given by $V(x, y) = 4e^{2x} + f(x) - 3y^2$. Given that the x -component of the electric field, E_x , and V are zero at the origin, $f(x)$ is:

(a) $3x^2 - 4e^{2x} + 8x$ (b) $3x^2 - 4e^{2x} + 16x$ (c) $4e^{2x} - 8$ (d) $3x^2 - 4e^{2x}$

Soln. In a region where charge density ρ is zero, the electrostatic potential V satisfies Laplace equation.

$\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \dots (i)$

$$V(x, y) = 4e^{2x} + f(x) - 3y^2$$

$$\text{Now, } \frac{\partial V}{\partial x} = 8e^{2x} + f'(x) \Rightarrow \frac{\partial^2 V}{\partial x^2} = 16e^{2x} + f''(x)$$

$$\frac{\partial V}{\partial y} = -6y \Rightarrow \frac{\partial^2 V}{\partial y^2} = -6$$

Using in equation (i), we get

$$16e^{2x} + f''(x) - 6 = 0 \Rightarrow f''(x) = 6 - 16e^{2x} \quad \dots (ii)$$

$$\text{Now, } E_x = -\frac{\partial V}{\partial x} = 0 \Rightarrow -E_x = \frac{\partial V}{\partial x} = 8e^{2x} + f'(x)$$

Using, $E_x(x=0) = 8 + f'(0) = 0$, we get $f'(0) = -8$

Using, $V(x, y) = 4e^{2x} + f(x) - 3y^2$ is zero at origin $(0, 0)$. So, $f(0) = -4$

On integrating $f''(x)$ from equation (ii), we get

$$f'(x) = 6x - 8e^{2x} + c \Rightarrow f'(0) = -8 + c \Rightarrow c = 0$$

$$\text{On further integration, } f(x) = 3x^2 - 4e^{2x} + D \Rightarrow f(0) = -4 + D \Rightarrow D = 0$$

$$\text{So, } f(x) = 3x^2 - 4e^{2x}$$

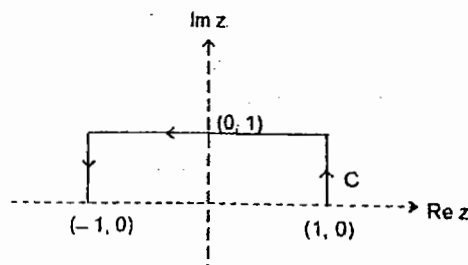
Correct answer: (d)

24. Consider the transition of liquid water to steam as water boils at a temperature of 100°C under a pressure of 1 atmosphere. Which one of the following quantities does not change discontinuously at the transition?

- (a) The Gibbs free energy (b) The internal energy
(c) The entropy (d) The specific volume

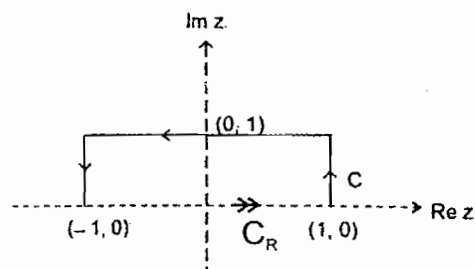
Soln. Correct answer : (a)

25. The value of the integral $\int_C dz z^2 e^z$, where C is an open contour in the complex z -plane as shown in the figure below, is:



- (a) $\frac{5}{e} + e$ (b) $e - \frac{5}{e}$ (c) $\frac{5}{e} - e$ (d) $-\frac{5}{e} - e$

Soln.



Since $z^2 e^z$ is an analytic function, then $\oint_{C_R+C} z^2 e^z dz = 0 \Rightarrow \int_C z^2 e^z dz = - \int_{C_R} z^2 e^z dz$

Then, value of open contour of integral $= - \int_{-1}^1 x^2 e^x dx$

$$\begin{aligned} I &= \int_{-1}^1 x^2 e^x dx = x^2 e^x \Big|_{-1}^{+1} - \int_{-1}^{+1} e^x 2x dx = (e^1 - e^{-1}) - 2 \left[\int_{-1}^1 x e^x dx \right] = (e^1 - e^{-1}) - 2 \left[x e^x \Big|_{-1}^1 - \int_{-1}^1 e^x dx \right] \\ &= (e^1 - e^{-1}) - 2 \left[e^1 + e^{-1} - \left[e^x \right]_{-1}^1 \right] = e - \frac{5}{e} \\ \therefore \text{Value of open integral} &= -e + \frac{5}{e} = \frac{5}{e} - e \end{aligned}$$

Correct answer : (c)

26. Which of the following matrices is an element of the group $SU(2)$?

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$

(c) $\begin{pmatrix} 2+i & 1 \\ 3 & 1+i \end{pmatrix}$

(d) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{2}{\sqrt{3}} & \frac{1}{2} \end{pmatrix}$

Soln. A matrix $U(a, b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, with $\det(U) = 1$ is called an element of the special unitary 2×2 group $SU(2)$.

The matrix $\begin{pmatrix} \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$ belongs to $SU(2)$.

$$a = \frac{1+i}{\sqrt{3}}, \quad b = -\frac{1}{\sqrt{3}}; \quad a^* = \frac{1-i}{\sqrt{3}}, \quad -b^* = \frac{1}{\sqrt{3}}$$

$$\text{Determinant} = aa^* + bb^* = |a|^2 + |b|^2 = \frac{|1+i|^2}{3} + \frac{1}{3} = \frac{1+1}{3} + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1$$

Correct answer : (b)

27. For constant uniform electric and magnetic fields $\vec{E} = \vec{E}_0$ and $\vec{B} = \vec{B}_0$, it is possible to choose a gauge such that the scalar potential ϕ and vector potential $\vec{A}$ are given by

- (a) $\phi = 0$ and $\vec{A} = \frac{1}{2}(\vec{B}_0 \times \vec{r})$ (b) $\phi = -\vec{E}_0 \cdot \vec{r}$ and $\vec{A} = \frac{1}{2}(\vec{B}_0 \times \vec{r})$
 (c) $\phi = -\vec{E}_0 \cdot \vec{r}$ and $\vec{A} = 0$ (d) $\phi = 0$ and $\vec{A} = -\vec{E}_0 t$

Soln. $\vec{B} = \vec{\nabla} \times \vec{A}$ Choose $\vec{A} = \frac{1}{2}(\vec{B}_0 \times \vec{r})$, $\vec{B}_0 = \text{constant magnetic field}$

$$\begin{aligned}\vec{\nabla} \times \vec{A} &= \frac{1}{2}[\vec{\nabla} \times (\vec{B}_0 \times \vec{r})] \\ &= \frac{1}{2}[-(\vec{B}_0 \cdot \vec{\nabla})\vec{r} + (\vec{r} \cdot \vec{\nabla})\vec{B}_0 + \vec{B}_0(\vec{\nabla} \cdot \vec{r}) - \vec{r}(\vec{\nabla} \cdot \vec{B}_0)]\end{aligned}$$

[Vector identity : $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$]

$$(\vec{B}_0 \cdot \vec{\nabla})\vec{r} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) (\hat{i}x + \hat{j}y + \hat{k}z) = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z = \vec{B}_0$$

$$\begin{aligned}(\vec{r} \cdot \vec{\nabla})\vec{B}_0 &= \left[(\hat{i}x + \hat{j}y + \hat{k}z) \cdot \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \right] \vec{B}_0 \\ &= \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = \hat{i}x \frac{\partial B_x}{\partial x} + \hat{j}y \frac{\partial B_y}{\partial y} + \hat{k}z \frac{\partial B_z}{\partial z} = 0 \\ &\quad [\because B_0 \text{ is independent of } x, y, z]\end{aligned}$$

$$\vec{\nabla} \cdot \vec{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (\hat{i}x + \hat{j}y + \hat{k}z) = 3$$

$$\vec{\nabla} \cdot \vec{B}_0 = 0$$

$$\therefore \vec{\nabla} \times \vec{A} = \frac{1}{2}[3\vec{B}_0 - \vec{B}_0] = \vec{B}_0 \text{ (which is true)}$$

$$\text{Now, } \phi = -\vec{E}_0 \cdot \vec{r}, \quad \vec{\nabla} \phi = -\vec{\nabla}(\vec{E}_0 \cdot \vec{r})$$

$$\begin{aligned}&= -\vec{\nabla}[(\hat{i}E_x + \hat{j}E_y + \hat{k}E_z) \cdot (\hat{i}x + \hat{j}y + \hat{k}z)] \\ &= -\vec{\nabla}(xE_x + yE_y + zE_z) \\ &= -\left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] [xE_x + yE_y + zE_z] \\ &= -\hat{i}E_x - \hat{j}E_y - \hat{k}E_z = -\vec{E}\end{aligned}$$

Correct answer : (b)

28. Let $\vec{a}$ and $\vec{b}$ be two distinct three-dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by

(a) $\frac{\vec{a} \times (\vec{b} \times \vec{a})}{a^2}$ (b) $\frac{\vec{b} \times (\vec{a} \times \vec{b})}{b^2}$ (c) $\frac{(\vec{a} \cdot \vec{b}) \vec{b}}{b^2}$ (d) $\frac{(\vec{b} \cdot \vec{a}) \vec{a}}{a^2}$

Soln. Component of $\vec{b}$ that is perpendicular to $\vec{a} = b \sin \theta$

Now, $\frac{\vec{a} \times (\vec{b} \times \vec{a})}{a^2} = \frac{\vec{a} \times (b \sin \theta \hat{n})}{a^2}$

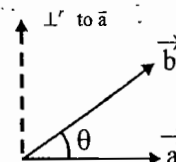
[where $\hat{n}$ is a unit vector perpendicular to $\vec{a}, \vec{b}$]

$$= \frac{ba \sin \theta (a \cdot 1 \cdot \sin 90^\circ) \hat{p}}{a^2}$$

[where $\hat{p}$ is a unit vector perpendicular to $\vec{a}, \hat{n}$]

$$= b \sin \theta \hat{p}$$

Correct answer : (a)



29. The wavefunction of a particle is given by $\psi = \left(\frac{1}{\sqrt{2}} \phi_0 + i \phi_1 \right)$, where ϕ_0 and ϕ_1 are the normalized eigenfunctions with energies E_0 and E_1 corresponding to the ground state and first excited state, respectively. The expectation value of the Hamiltonian in the state ψ is:

(a) $\frac{E_0}{2} + E_1$ (b) $\frac{E_0}{2} - E_1$ (c) $\frac{E_0 + 2E_1}{3}$ (d) $\frac{E_0 + 2E_1}{3}$

Soln. $|\psi\rangle = \frac{1}{\sqrt{2}} |\phi_0\rangle + i |\phi_1\rangle$, $\Rightarrow \langle\psi| = \frac{1}{\sqrt{2}} \langle\phi_0| - i \langle\phi_1|$

Given, $H|\phi_0\rangle = E_0 |\phi_0\rangle$ and $H|\phi_1\rangle = E_1 |\phi_1\rangle$

$$\langle H \rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$= \frac{\left(\frac{1}{\sqrt{2}} \langle\phi_0| - i \langle\phi_1| \right) H \left(\frac{1}{\sqrt{2}} |\phi_0\rangle + i |\phi_1\rangle \right)}{\left(\frac{1}{\sqrt{2}} \langle\phi_0| - i \langle\phi_1| \right) \left(\frac{1}{\sqrt{2}} |\phi_0\rangle + i |\phi_1\rangle \right)} = \frac{\left(\frac{1}{\sqrt{2}} \langle\phi_0| - i \langle\phi_1| \right) \left(\frac{E_0}{\sqrt{2}} |\phi_0\rangle + i E_1 |\phi_1\rangle \right)}{\frac{1}{2} + 1}$$

$$= \frac{\left(\frac{E_0}{2} + E_1 \right)}{\frac{3}{2}} = \frac{E_0 + 2E_1}{3}$$

Correct answer : (d)

30. A particle is confined to the region $x \geq 0$ by a potential which increases linearly as $u(x) = u_0 x$. The mean position of the particle at temperature T is:

(a) $\frac{k_B T}{u_0}$ (b) $(k_B T)^2 / u_0$ (c) $\sqrt{\frac{k_B T}{u_0}}$ (d) $u_0 k_B T$

Soln. If potential varies as x^{n+1} . Then according to virial Theorem.

$$\bar{T} = \frac{(n+1)}{2} \bar{V}$$

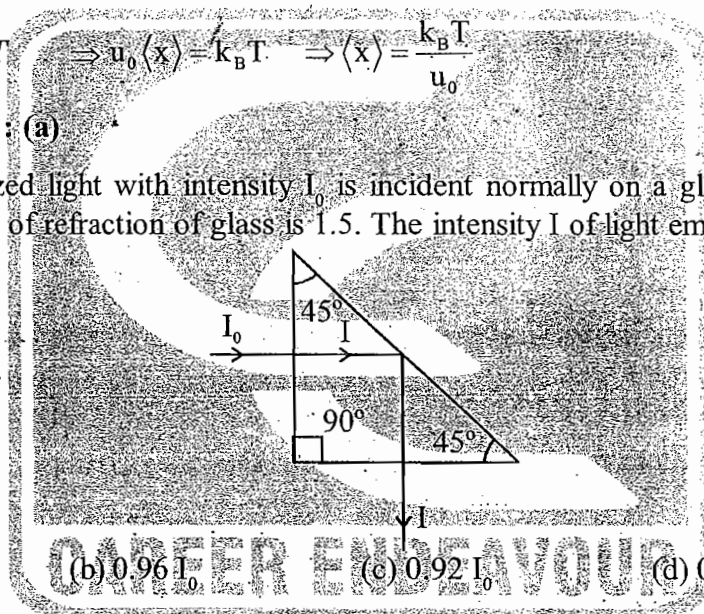
For given potential i.e. $u(x) = u_0 x$, $\bar{T} = \frac{1}{2} \bar{V}$

According to equipartition Theorem, $\bar{E} = \frac{3}{2} k_B T$

So, then $\bar{V} = k_B T \Rightarrow u_0 \langle x \rangle = k_B T \Rightarrow \langle x \rangle = \frac{k_B T}{u_0}$

Correct Answer : (a)

31. Circularly polarized light with intensity I_0 is incident normally on a glass prism as shown in the figure. The index of refraction of glass is 1.5. The intensity I of light emerging from the prism is:



(a) I_0 (b) $0.96 I_0$ (c) $0.92 I_0$ (d) $0.88 I_0$

Soln. The right angle prism acts as a total internal reflector.

Normal incidence:

Coefficient of reflection : $R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$

Coefficient of transition: $T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$

For $n_1 = 1.5, n_2 = 1.0 \Rightarrow T = \frac{4 \times 1.5}{(1+1.5)^2} = \frac{6}{(2.5)^2} = \frac{6.00}{6.25} = 0.96$

Intensity of emerging light = $I = (0.96)^2 I_0$

The critical angle (θ_c) for glass – air interface is 42° . So, wave inside the prism will be total internally reflected

Correct answer : (c).

32. The acceleration due to gravity (g) on the surface of Earth is approximately 2.6 times that on the surface of Mars. Given that the radius of Mars is about one half the radius of Earth, the ratio of the escape velocity on Earth to that on Mars is approximately.
- (a) 1.1 (b) 1.3 (c) 2.3 (d) 5.2

Soln. Escape velocity $v = \sqrt{\frac{2GM}{r}} = \sqrt{2gr}$

$$\frac{v_{\text{earth}}}{v_{\text{mars}}} = \sqrt{\frac{2g_{\text{earth}}r_{\text{earth}}}{2g_{\text{mars}}r_{\text{mars}}}}; \text{ Since, } \frac{g_{\text{earth}}}{g_{\text{mars}}} = 2.6 \text{ and } \frac{r_{\text{earth}}}{r_{\text{mars}}} = 2$$

Then, $\frac{v_{\text{earth}}}{v_{\text{mars}}} = \sqrt{2.6 \times 2} = \sqrt{5.2} = 2.28 \approx 2.3$

Correct answer: (c)

33. A plane electromagnetic wave is propagating in a lossless dielectric. The electric field is given by $E(x, y, z, t) = E_0(\hat{x} + A\hat{z})\exp\left[ik_0\left\{-ct + (x + \sqrt{3}z)\right\}\right]$, where c is the speed of light in vacuum, E_0 , A and k_0 are constants and $\hat{x}$ and $\hat{z}$ are unit vectors along the x - and z -axes. The relative dielectric constant of the medium, ϵ_r , and the constant A are

(a) $\epsilon_r = 4$ and $A = -\frac{1}{\sqrt{3}}$

(b) $\epsilon_r = 4$ and $A = +\frac{1}{\sqrt{3}}$

(c) $\epsilon_r = 4$ and $A = \sqrt{3}$

(d) $\epsilon_r = 4$ and $A = -\sqrt{3}$

Soln. $\vec{E} = \vec{E}_0 \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$ (1) (General form)

$E(x, y, z, t) = E_0(\hat{x} + A\hat{z})\exp\left[i\left\{-k_0ct + k_0(x + \sqrt{3}z)\right\}\right]$... (2)

Comparing (1) and (2) wave vector

$\vec{k} = k_0(\hat{x} + \sqrt{3}\hat{z})$

$\Rightarrow |\vec{k}| = 2k_0 \text{ and } \omega = k_0c$

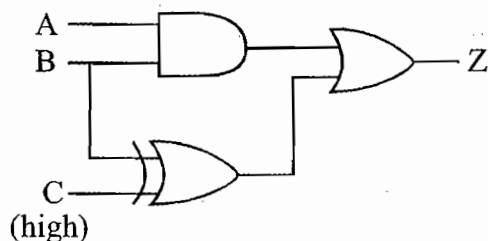
Wave velocity $v = \frac{\omega}{|\vec{k}|} = \frac{c}{2}$, We also know $v = \frac{c}{n} \Rightarrow n = 2$

Relative dielectric constant for the medium $\epsilon_r = n^2 = 4$

Since, $\vec{E} \cdot \vec{k} = 0 \Rightarrow A = -\frac{1}{\sqrt{3}}$

Correct answer: (a)

34. Consider the digital circuit shown below in which the input C is always high (I).



The truth table for the circuit can be written as

A	B	Z
0	0	
0	1	
1	0	
1	1	

The entries in the Z column (vertically) are

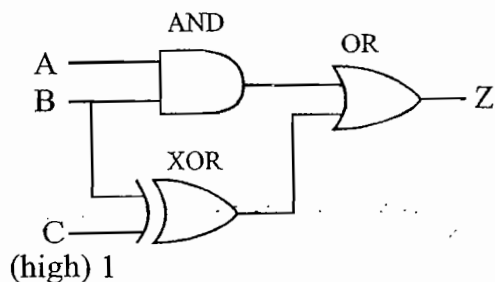
(a) 1010

(b) 0100

(c) 1111

(d) 1011

Soln.

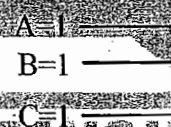
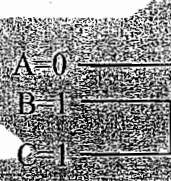
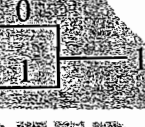
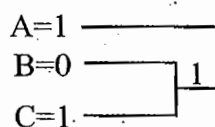
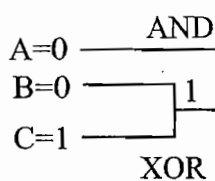


Truth Table:

AND		
A	B	Q = AB
0	0	0
0	1	0
1	0	0
1	1	1

OR		
A	B	Q = A + B
0	0	0
0	1	1
1	0	1
1	1	1

XOR		
A	B	Q = (A + B)
0	0	0
0	1	1
1	0	1
1	1	0



A	B	Z
0	0	1
0	1	0
1	0	1
1	1	1

Correct answer : (d)

35. The energy levels of the non-relativistic electron in a hydrogen atom (i.e. in a Coulomb potential $V(r) \propto -1/r$) are given by $E_{n\ell m} \propto -1/n^2$, where n is the principal quantum number, and the corresponding wave functions are given by $\psi_{n\ell m}$, where ℓ is the orbital angular momentum quantum number and m is the magnetic quantum number. The spin of the electron is not considered. Which of the following is a correct statement?
- (a) There are exactly $(2\ell + 1)$ different wave functions $\psi_{n\ell m}$, for each $E_{n\ell m}$.
- (b) There are $\ell(\ell + 1)$ different wave functions $\psi_{n\ell m}$, for each $E_{n\ell m}$.

(c) $E_{n\ell m}$ does not depend on ℓ and m for the Coulomb potential.

(d) There is a unique wave function $\psi_{n\ell m}$ and $E_{n\ell m}$.

Soln. $E_n = -\frac{13.6 \text{ eV}}{n^2}$

The energy E_n depends only on n and not on ℓ and m . This is due to the degeneracy n^2 (neglecting spin) that arises due to the spherical symmetry of the Coulomb potential which depends only on radial distance 'r' of the electron from the nucleus which is considered as the origin. There is no angular dependence (θ and ϕ) in $V(r)$.

Correct answer: (c).

36. The Hamiltonian of an electron in a constant magnetic field $\vec{B}$ is given by $H = \mu \vec{\sigma} \cdot \vec{B}$ where μ is a positive constant and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli matrices. Let $\omega = \mu B / \hbar$ and I be the 2×2 unit matrix. Then the operator $e^{iHt/\hbar}$ simplifies to

- (a) $I \cos \frac{\omega t}{2} + \frac{i \vec{\sigma} \cdot \vec{B}}{B} \sin \frac{\omega t}{2}$ (b) $I \cos \omega t + \frac{i \vec{\sigma} \cdot \vec{B}}{B} \sin \omega t$
(c) $I \sin \omega t + \frac{i \vec{\sigma} \cdot \vec{B}}{B} \cos \omega t$ (d) $I \sin 2\omega t + \frac{i \vec{\sigma} \cdot \vec{B}}{B} \cos 2\omega t$

Soln. We know that,

$$e^{i(\vec{\sigma} \cdot \hat{n})\theta} = I \cos \theta + i(\vec{\sigma} \cdot \hat{n}) \sin \theta$$

$$\begin{aligned} \text{Now, } e^{iHt/\hbar} &= e^{i(\mu \vec{\sigma} \cdot \vec{B})t/\hbar} = e^{i(\vec{\sigma} \cdot \vec{B})Bt/\hbar} = e^{i(\vec{\sigma} \cdot \vec{B})\omega t} \\ &= I \cos \omega t + i(\vec{\sigma} \cdot \vec{B}) \sin \omega t \\ &= I \cos \omega t + \frac{i(\vec{\sigma} \cdot \vec{B})}{B} \sin \omega t \end{aligned}$$

Correct answer: (b)

37. The Hamiltonian of a system with n degrees of freedom is given by

$$H(q_1, \dots, q_n; p_1, \dots, p_n; t),$$

with an explicit dependence on the time t . Which of the following is correct?

- (a) Different phase trajectories cannot intersect each other
(b) H always represents the total energy of the system and is a constant of the motion.
(c) The equations $\dot{q}_i = \partial H / \partial p_i$, $\dot{p}_i = -\partial H / \partial q_i$ are not valid since H has explicit time dependence.
(d) Any initial volume element in phase space remains unchanged in magnitude under time evolution.

Soln. According to Liouville's theorem, the phase volume occupied by a collection of system evolve according to Hamilton's equation of motion, and will be preserved in time.

Correct answer: (d)

38. If the perturbation $H' = ax$, where a is a constant; is added to the infinite square well potential

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \pi \\ \infty & \text{otherwise} \end{cases}$$

The first order correction to ground state energy is:

- (a) $\frac{a\pi}{2}$ (b) $a\pi$ (c) $\frac{a\pi}{4}$ (d) $\frac{a\pi}{\sqrt{2}}$

Soln. For a square well potential of infinite height at 0 and π .

Ground State function, $\psi_0(x) = \sqrt{\frac{2}{\pi}} \sin x$

general form of wave function, $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \dots$

First-order correction to the ground state is:

$$\begin{aligned} \Delta E &= \langle \psi_0 | H' | \psi_0 \rangle \\ &= \int_0^\pi \left(\frac{2}{\pi} \sin^2 x \right) ax \, dx = \frac{2a}{\pi} \int_0^\pi x (\sin^2 x) \, dx = \frac{2a}{\pi} \int_0^\pi x \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \frac{a}{\pi} \left[\int_0^\pi x \, dx - \int_0^\pi x \cos 2x \, dx \right] = \frac{a}{\pi} \left[\frac{x^2}{2} \Big|_0^\pi - \left\{ x \frac{\sin 2x}{2} \Big|_0^\pi - \int_0^\pi \frac{\sin 2x}{2} dx \right\} \right] \\ &= \frac{a}{\pi} \left[\frac{\pi^2}{2} - \left\{ 0 - \frac{1}{2} \frac{\cos 2x}{2} \Big|_0^\pi \right\} \right] = \frac{a}{\pi} \left[\frac{\pi^2}{2} + \frac{1}{4} (\cos 2\pi - 1) \right] = \frac{a\pi}{2} \end{aligned}$$

Correct answer : (a).

39. Let $p_n(x)$ (where $n = 0, 1, 2, \dots$) be a polynomial of degree n with real coefficients, defined in the

interval $2 \leq x \leq 4$. If $\int_2^4 p_n(x) p_m(x) dx = \delta_{nm}$, then

- (a) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(-3-x)$ (b) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{3}(3+x)$
 (c) $p_0(x) = \frac{1}{2}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$ (d) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

Soln. According to the given orthonormality condition,

$$(i) \int_2^4 P_0(x) \cdot P_0(x) dx = 1 \quad (ii) \int_2^4 P_0(x) P_1(x) dx = 0 \quad (iii) \int_2^4 P_1(x) P_2(x) dx = 1$$

Only option (d) satisfies all the above equation.

Correct answer: (d)

40. A cavity contains blackbody radiation in equilibrium at temperature T . The specific heat per unit volume of the photon gas in the cavity is of the form $C_V = \gamma T^3$ where γ is a constant. The cavity is expanded to twice its original volume and then allowed to equilibrate at the same temperature T . The new internal energy per unit volume is:

- (a) $4\gamma T^4$ (b) $2\gamma T^4$ (c) $\frac{\gamma T^4}{4}$ (d) γT^4

Soln. For Blackbody radiation, $C_V = \frac{16\sigma}{c} VT^3$ where, σ is Stefan-Boltzman constant.

Put $\gamma = \frac{16\sigma}{c}$ $\therefore C_V = \gamma VT^3$

Internal energy, $E = \int C_V dT = \gamma V \int T^3 dT = \frac{\gamma VT^4}{4}$

When volume is doubled $C_V = 2\gamma VT^3$

New internal energy $= 2\gamma V \int T^3 dT = \frac{2\gamma VT^4}{4} = \frac{\gamma VT^4}{2}$

Change in energy $= \frac{\gamma VT^4}{2} - \frac{\gamma VT^4}{4} = \frac{\gamma VT^4}{4}$

Change in energy/volume $= \frac{\gamma T^4}{4}$

Correct answer : (d)

PART - C

41. Consider a system of N non-interacting spins, each of which has classical magnetic moment of magnitude μ . The Hamiltonian of this system in an external magnetic field $\vec{H}$ is $H = -\sum_{i=1}^N \vec{\mu}_i \cdot \vec{H}$, where $\vec{\mu}_i$ is the magnetic moment of the i^{th} spin. The magnetization per spin at temperature T is:

- (a) $\frac{\mu^2 H}{k_B T}$ (b) $\mu \left[\coth \left(\frac{\mu H}{k_B T} \right) - \frac{k_B T}{\mu H} \right]$
(c) $\mu \sinh \left(\frac{\mu H}{k_B T} \right)$ (d) $\mu \tanh \left(\frac{\mu H}{k_B T} \right)$

Soln. Interaction energy $= -\vec{\mu} \cdot \vec{H}$

$$\langle \mu_z \rangle = \frac{\int \mu_z e^{+\vec{\mu} \cdot \vec{H}/kT} d\Omega}{\int e^{+\vec{\mu} \cdot \vec{H}/kT} d\Omega}$$

$$\langle \mu_z \rangle = \frac{\mu \int_0^{2\pi} \int_0^\pi \cos \theta e^{\mu H \cos \theta / kT} \sin \theta d\theta d\phi}{\int_0^{2\pi} \int_0^\pi e^{\mu H \cos \theta / kT} \sin \theta d\theta d\phi} ; \text{ [Using } \int d\phi = 2\pi \text{ and put } \cos \theta = x, \mu H / kT = a \text{]}$$

$$\langle \mu_z \rangle = \frac{\mu \int_0^\pi \cos \theta e^{a \cos \theta} \sin \theta d\theta}{\int_0^\pi e^{a \cos \theta} \sin \theta d\theta} = \mu \left[\frac{\int_{-1}^{+1} x e^{ax} dx}{\int_{-1}^{+1} e^{ax} dx} \right]$$

$$\text{First integral} = \mu \left[x \frac{e^{ax}}{a} \Big|_{-1}^{+1} - \frac{1}{a} \int_{-1}^{+1} e^{ax} dx \right] = \mu \left[\left(\frac{e^a + e^{-a}}{2a} \right) - \frac{1}{a} \left(\frac{e^{ax}}{a} \Big|_{-1}^{+1} \right) \right]$$

$$= \mu \left[\frac{2 \cosh a}{a} - \frac{1}{a^2} \sinh a \right] = \frac{2\mu}{a} \left[\cosh a - \frac{\sinh a}{a} \right]$$

$$\text{Second integral} = \int_{-1}^{+1} e^{ax} dx = \frac{2 \sinh a}{a}$$

$$\langle \mu_z \rangle = \frac{\mu \frac{2}{a} \left(\cosh a - \frac{1}{a} \sinh a \right)}{\frac{2 \sinh a}{a}} = \mu \left(\coth a - \frac{1}{a} \right)$$

$$\langle \mu_z \rangle = \mu \left(\coth \frac{\mu H}{kT} - \frac{kT}{\mu H} \right)$$

Correct answer : (b)

42. Which of the following is an analytic function of the complex variable $z = x + iy$ in the domain $|z| < 2$?

- (a) $(3 + x - iy)^7$ (b) $(1 + x + iy)^4 (7 - x - iy)^3$
 (c) $(1 - 2x - iy)^4 (3 - x - iy)^3$ (d) $(x + iy - 1)^{1/2}$

Soln. Any complex-function which contains only 'z', will be analytic in nature.

But any complex function which contains $\bar{z}$, $|z|$, $\text{Re}.z$, $\text{Im}.z$, will not be analytic in nature.

Option-1 : $(3 + x - iy)^3 = (3 + \bar{z})^3$

Option-2 : $(1 + x + iy)^4 (7 - x - iy)^3 = (1 + z)^4 (7 - z)^3$

Option-3 : $(1 - 2x - iy)^4 (3 - x - iy)^3 = (1 - \text{Re}.z - z)^4 (3 - z)^3$

Option-4 : $(x + iy - 1)^{1/2} = (z - 1)^{1/2}$ has a branch point at $z = 1$

Correct answer : (b)

43. A particle in one dimension moves under the influence of a potential $V(x) = ax^6$, where a is a real constant. For large n the quantized energy level E_n depends on n as:

- (a) $E_n \sim n^3$ (b) $E_n \sim n^{4/3}$ (c) $E_n \sim n^{6/5}$ (d) $E_n \sim n^{3/2}$

Soln. $V(x) = ax^6$

For classical turning points, $E = V(x) = ax^6$

$$\Rightarrow x = \pm \sqrt[6]{E/a} = \pm \left(\frac{E}{a}\right)^{1/6} \Rightarrow x = \pm b \text{ (say)}$$

Using quantization condition of WKB approximation,

$$\int_{-b}^x p(x) dx + \frac{\pi}{4} + \int_x^b p(x) dx + \frac{\pi}{4} = n\pi$$

$$\Rightarrow \int_{-b}^b \frac{2m}{\hbar^2} \sqrt{E - V(x)} dx = (n-1) \frac{\pi}{2}$$

$$\Rightarrow \frac{2m}{\hbar^2} \sqrt{E} \int_{-b}^b \sqrt{1 - \frac{V(x)}{E}} dx = (n-1) \frac{\pi}{2}$$

Using dimensional analysis in terms of energy.

$$E^{1/2} \cdot E^{1/6} \propto n \Rightarrow E \propto n^{3/2}$$

Correct answer : (d).

44. The Lagrangian of a particle of charge e and mass m in applied electric and magnetic fields is given by $L = \frac{1}{2} m \vec{v}^2 + e \vec{A} \cdot \vec{v} - e\phi$, where $\vec{A}$ and ϕ are the vector and scalar potentials corresponding to the magnetic and electric fields, respectively. Which of the following statements is correct?

- (a) The canonically conjugate momentum of the particle is given by $\vec{p} = m\vec{v}$
 (b) The Hamiltonian of the particle is given by $H = \frac{\vec{p}^2}{2m} + \frac{e}{m} \vec{A} \cdot \vec{p} + e\phi$
 (c) L remains unchanged under a gauge transformation of the potentials.
 (d) Under a gauge transformation of the potentials, L changes by the total time derivative of a function of $\vec{r}$ and t .

Soln. $L = \frac{1}{2} m \vec{v}^2 - e\phi + e \vec{A} \cdot \vec{v}$

We know that $\vec{E}$ and $\vec{B}$ fields are invariant under gauge transformation.

$$\vec{A}(\vec{x}, t) \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \lambda(\vec{x}, t); \quad \phi(\vec{x}, t) \rightarrow \phi' = \phi - \frac{\partial \lambda}{\partial t}(\vec{x}, t)$$

where $\lambda(\vec{x}, t)$ is an arbitrary scalar function.

$$\therefore L \rightarrow L' = L + e \left(\frac{\partial \lambda}{\partial t}(\vec{x}, t) + \vec{v} \cdot \vec{\nabla} \lambda(\vec{x}, t) \right)$$

The expression in bracket is just the total time derivative of $\lambda(\vec{x}, t)$.

If we add a total time derivative of a function of $\vec{x}$ and t to the Lagrangian, the equations of motion do not change.

Correct answer : (d)

45. A static, spherically symmetric charge distribution is given by $\rho(r) = \frac{A}{r} e^{-Kr}$ where A and K are positive constants. The electrostatic potential corresponding to this charge distribution varies with r as

(a) $r e^{-Kr}$ (b) $\frac{1}{r} e^{-Kr}$ (c) $\frac{1}{r^2} e^{-Kr}$ (d) $\frac{1}{r} (1 - e^{-Kr})$

Soln. This is a spherically symmetric charge distribution.
From Gauss Law, we get

$$\begin{aligned}
 E(r) &= \frac{1}{4\pi\epsilon_0 r^2} \int_0^r \rho(r) 4\pi r^2 dr \\
 &= \frac{1}{4\pi\epsilon_0 r^2} \int_0^r \frac{A e^{-Kr}}{r} 4\pi r^2 dr = \frac{A}{\epsilon_0 r^2} \int_0^r r e^{-Kr} dr \\
 E(r) &= \frac{A}{\epsilon_0 r^2} \left[\frac{r}{-K} e^{-Kr} - \frac{1}{K^2} (e^{-Kr} - 1) \right] \\
 \therefore V(r) &= \int_r^\infty E(r) dr = \frac{A}{\epsilon_0} \left[-\frac{1}{K} \int_r^\infty \frac{e^{-Kr}}{r} dr - \frac{1}{K^2} \int_r^\infty \frac{e^{-Kr} - 1}{r^2} dr \right] \\
 &= \frac{A}{\epsilon_0} \left[-\frac{1}{K} \left[\frac{e^{-Kr}}{r} \right]_r^\infty + \frac{1}{K^2} \int_r^\infty \frac{e^{-Kr}}{r^2} dr - \frac{1}{K^2} \int_r^\infty \frac{1}{r^2} dr \right] \\
 &= \frac{A}{\epsilon_0} \left[\frac{1}{K^2} \left(0 - \frac{e^{-Kr}}{r} \right) + \frac{1}{K^2} \int_r^\infty \frac{dr}{r^2} \right] \\
 &= \frac{A}{\epsilon_0} \left[\frac{-1 e^{-Kr}}{K^2 r} + \frac{1}{K^2} \frac{1}{r} \right] = \frac{A}{\epsilon_0 K^2} \left[\frac{1 - e^{-Kr}}{r} \right]
 \end{aligned}$$

Correct answer: (d)

46. Consider two independently diffusing non-interacting particles in 3-dimensional space, both placed at the origin at time $t = 0$. These particles have different diffusion constants D_1 and D_2 . The quantity $\langle [\bar{R}_1(t) - \bar{R}_2(t)]^2 \rangle$ where $\bar{R}_1(t)$ and $\bar{R}_2(t)$ are the positions of the particles at time t, behaves as:

(a) $6t(D_1 + D_2)$ (b) $6t(D_1 - D_2)$ (c) $6t\sqrt{D_1^2 + D_2^2}$ (d) $6t\sqrt{D_1 D_2}$

Soln. Diffusion equation in one-dimension for one particle is

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

with $\rho(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$ as solution by assuming $x = 0$ at $t = 0$.

For this Gaussian distribution, $\overline{x^2} = 2Dt$

In 3-D, it yields $\langle R_1^2(t) \rangle = 6D_1t$, where $\rho(\vec{R}_1, t) \propto e^{-R_1^2/6D_1t}$

For second particle $\rho(\vec{R}_2, t) \propto e^{-R_2^2/6D_2t}$, then $\langle R_2^2(t) \rangle = 6D_2t$

$$\begin{aligned} \langle [\vec{R}_1(t) - \vec{R}_2(t)]^2 \rangle &= \langle R_1^2(t) \rangle + \langle R_2^2(t) \rangle - 2\langle R_1(t) \cdot R_2(t) \rangle \\ &= \langle R_1^2(t) \rangle + \langle R_2^2(t) \rangle - 2\langle R_1(t) \rangle \langle R_2(t) \rangle \quad [\text{Since, } \langle R_1(t) \rangle = \langle R_2(t) \rangle = 0] \end{aligned}$$

$$\text{So, } \langle [\vec{R}_1(t) - \vec{R}_2(t)]^2 \rangle = 6t(D_1 + D_2)$$

Correct answer : (a)

47. A resistance is measured by passing current through it and measuring the resulting voltage drop. If the voltmeter and the ammeter have uncertainties of 3% and 4%, respectively, then

(A) The uncertainty in the value of resistance is:

- (a) 7.0 % (b) 3.5 % (c) 5.0 % (d) 12.0 %

(B) The uncertainty in the computed value of the power dissipated in resistance is

- (a) 7% (b) 5% (c) 11% (d) 9%

Soln. (A) $R = \frac{V}{I}$

Uncertainties in V and I are independent. So, they must add in quadrature

$$\% \text{ error in } R = \frac{\Delta R}{R} \times 100 = \sqrt{\left(\frac{\Delta V}{V} \times 100\right)^2 + \left(\frac{\Delta I}{I} \times 100\right)^2} = \sqrt{9 + 16} = \sqrt{25} = 5\%$$

Correct answer : (c)

(B) Power Dissipated $P = VI$

$$\text{Again, } \% \text{ error in } P = \frac{\Delta P}{P} \times 100 = \sqrt{\left(\frac{\Delta V}{V} \times 100\right)^2 + \left(\frac{\Delta I}{I} \times 100\right)^2} = \sqrt{9 + 16} = \sqrt{25} = 5\%$$

Correct answer : (b)

48. In the absence of an applied torque a rigid body with three distinct principal moments of inertia given by I_1, I_2 and I_3 is rotating freely about a fixed point inside the body. The Euler equations for the components of its angular velocity $(\omega_1, \omega_2, \omega_3)$ are

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3, \quad \dot{\omega}_2 = \frac{I_3 - I_1}{I_2} \omega_1 \omega_3, \quad \dot{\omega}_3 = \frac{I_1 - I_2}{I_3} \omega_1 \omega_2$$

(A) The equilibrium points in $(\omega_1, \omega_2, \omega_3)$ space are

- (a) $(1, -1, 0), (-1, 0, 1)$ and $(0, -1, 1)$ (b) $(1, 1, 0), (1, 0, 1)$ and $(0, 1, 1)$
(c) $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ (d) $(1, 1, 1), (-1, -1, -1)$ and $(0, 0, 0)$

(B) The constants of motion are

- (a) $\omega_1^2 + \omega_2^2 + \omega_3^2$ and $I_1\omega_1 + I_2\omega_2 + I_3\omega_3$ (b) $I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$ and $I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$
 (c) $I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$ and $\omega_1 + \omega_2 + \omega_3$ (d) $\omega_1^2 + \omega_2^2 + \omega_3^2$ and $I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$

Soln. (A) Euler equations:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3, \quad \dot{\omega}_2 = \frac{I_3 - I_1}{I_2} \omega_1 \omega_3, \quad \dot{\omega}_3 = \frac{I_1 - I_2}{I_3} \omega_1 \omega_2$$

Equilibrium points $\Rightarrow \dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

Then, $\omega_2 \omega_3 = 0$; $\omega_1 \omega_3 = 0$; $\omega_1 \omega_2 = 0$

The general solution is $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ for any value of a, b and c.

Any point on the co-ordinate axes is an equilibrium point. choose $a = b = c = 1$.

Correct answer : (c)

(B) Torque, $\vec{\tau} = \frac{d\vec{L}}{dt} = 0$ (No torque is applied)

L is a constant of motion.

$L = I\omega \Rightarrow L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$ (Inertial ellipsoid) is a constant of motion

Kinetic energy $T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$ is constant of motion

$\therefore 2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$ is constant.

Correct answer: (b)

49. In a system consisting of two spin $\frac{1}{2}$ particles labeled 1 and 2, let $\vec{S}^{(1)} = \frac{\hbar}{2}\vec{\sigma}^{(1)}$ and $\vec{S}^{(2)} = \frac{\hbar}{2}\vec{\sigma}^{(2)}$ denote the corresponding spin operators. Here $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ and $\sigma_x, \sigma_y, \sigma_z$ are the three Pauli matrices.

(A) In the standard basis the matrices for the operators $S_x^{(1)}S_y^{(2)}$ and $S_y^{(1)}S_x^{(2)}$ respectively,

- (a) $\frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{\hbar^2}{4} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\frac{\hbar^2}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\hbar^2}{4} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$
 (c) $\frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$ (d) $\frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(B) These two operators satisfy the relation

- (a) $\{S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\} = S_z^{(1)}S_z^{(2)}$ (b) $\{S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\} = 0$
 (c) $[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}] = iS_z^{(1)}S_z^{(2)}$ (d) $[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}] = 0$

Soln. (A) Pauli Spin matrices:

$$\sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x^{(1)} = I \otimes \sigma_x \quad (\text{Tensor Product})$$

[Multiply the second matrix by each element of the first matrix to get the block of matrix].

$$= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow S_y^{(2)} = \sigma_y \otimes I$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$S_x^{(1)} S_y^{(2)} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$S_y^{(1)} = I \otimes \sigma_y$$

$$= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$S_x^{(2)} = \sigma_x \otimes I = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_y^{(1)} S_x^{(2)} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Correct answer : (c)

$$(B) S_x^{(1)} S_y^{(2)} S_x^{(1)} S_y^{(2)} = S_x^{(1)} S_y^{(1)} S_y^{(2)} S_x^{(2)} = i \hbar S_z^{(1)} (-i \hbar S_z^{(2)}) = \hbar^2 S_z^{(1)} S_z^{(2)}$$

$$S_y^{(1)} S_x^{(2)} S_y^{(1)} S_x^{(2)} = S_y^{(1)} S_x^{(1)} S_x^{(2)} S_y^{(2)} = (-i \hbar) S_z^{(1)} (i \hbar S_z^{(2)}) = \hbar^2 S_z^{(1)} S_z^{(2)}$$

$$[S_x^{(1)} S_y^{(2)}, S_y^{(1)} S_x^{(2)}] = 0$$

Correct answer: (d)

50. Consider the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(A) The eigenvalues of M are

(a) 0, 1, 2

(b) 0, 0, 3

(c) 1, 1, 1

(d) -1, 1, 3

(B) The exponential of M simplifies to (I is the 3×3 identity matrix)

(a) $e^M = I + \left(\frac{e^3 - 1}{3} \right) M$

(b) $e^M = I + M + \frac{M^2}{2!}$

(c) $e^M = I + 3^3 M$

(d) $e^M = (e - 1)M$

Soln. (A) $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

For any square matrix M of order n having all elements equal to 1, have eigenvalues n, 0, 0, 0, ...
Therefore, eigen values of M are: 0, 0, 3

Correct answer : (b)

(B) $M^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3M$

$M^3 = M^2 \cdot M = 3M \cdot M = 3^2 M$ and so on.

$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \dots$

$= I + M + \frac{3M}{2!} + \frac{3^2 M}{3!} + \frac{3^3 M}{4!} + \dots$

$= I + \frac{M}{3} \left[3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \dots \right]$

$= I + \frac{M}{3} (e^3 - 1)$

Correct answer: (a)

51. The radius of a ${}^{64}_{29}\text{Cu}$ nucleus is measured to be 4.8×10^{-13} cm.

(A) The radius of a ${}^{27}_{12}\text{Mg}$ nucleus can be estimated to be

(a) 2.86×10^{-13} cm

(b) 5.2×10^{-13} cm

(c) 3.6×10^{-13} cm

(d) 8.6×10^{-13} cm

Soln. Since, $R = R_0(A)^{1/3}$

$$\text{So, } \frac{R_{Mg}}{R_{Cu}} = \left(\frac{A_{Mg}}{A_{Cu}} \right)^{1/3}$$

$$\Rightarrow R_{Mg} = \left(\frac{27}{64} \right)^{1/3} \times R_{Cu} = 3.6 \times 10^{-13} \text{ cm}$$

Correct answer : (c)

52. The character table of C_{3v} , the group of symmetries of an equilateral triangle is given below

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$
$1C_1$	1	1	b
$3C_2$	1	a	c
$2C_3$	1	1	d

In the above C_1, C_2, C_3 denotes the three classes of C_{3v} , containing 1, 3 and 2 elements respectively, and

$\chi^{(0)}, \chi^{(1)}$ and $\chi^{(2)}$ are the characters of the three irreducible representations $\Gamma^{(0)}, \Gamma^{(1)}$ and $\Gamma^{(2)}$ of C_{3v} .

(A) The entries a, b, c and d in this table are, respectively

- (a) 2, 1, -1, 0 (b) 1, 2, 0, -1 (c) -1, 1, 0, -1 (d) -1, 1, 1, -1

(B) The reducible representation Γ of C_{3v} with character $\chi = (4, 0, 1)$ decomposes into its irreducible representations $\Gamma^{(0)}, \Gamma^{(1)}, \Gamma^{(2)}$ as

- (a) $2\Gamma^{(0)} + 2\Gamma^{(1)}$ (b) $\Gamma^{(0)} + 3\Gamma^{(1)}$ (c) $\Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)}$ (d) $2\Gamma^{(2)}$

Soln. **(A)** The correct character table for C_{3v} point group is:

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$
$\Gamma^0 = 1C_1$	1	1	2
$\Gamma^1 = 3C_2$	1	-1	0
$\Gamma^2 = 2C_3$	1	1	-1

$$a = -1$$

$$b = 2$$

$$c = 0$$

$$d = -1$$

Order of group = Number of classes = $1 + 3 + 2 = 6 = R$

The entries satisfy great orthogonality theorem.

$$1(1)^2 + 3(1)^2 + 2(1)^2 = 6; 1(+1)^2 + 3(-1)^2 + 2(1)^2 = 6; 1(2)^2 + 3(0)^2 + 2(-1)^2 = 6$$

$$1(1)(1) + 3(1)(-1) + 2(1)(1) = 0; 1(1)(2) + 3(1)(0) + 2(1)(-1) = 0;$$

$$1(1)(2) + 3(-1)(0) + 2(1)(-1) = 0$$

Correct answer: (b)

(B) Number of classes $\chi^{(0)} \begin{Bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 4 & 0 & 1 \end{Bmatrix} \rightarrow \text{Multiply entries vertically}$

$$\Gamma \begin{Bmatrix} 4 & 0 & 2 \end{Bmatrix} \rightarrow \text{add and divide by } h = 6,$$

$$\text{Number of } \Gamma^0 = \frac{6}{6} = 1.$$

Number of Γ^1

$$\chi^{(1)} \begin{Bmatrix} 1 & 3 & 2 \\ 1 & -1 & 1 \\ 4 & 0 & 1 \end{Bmatrix}$$

$$\Gamma \begin{Bmatrix} 4 & 0 & 2 \end{Bmatrix}$$

Number of Γ^2

$$\chi^{(2)} \begin{Bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 4 & 0 & 1 \end{Bmatrix}$$

$$\Gamma \begin{Bmatrix} 8 & 0 & 2 \end{Bmatrix}$$

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)}$$

$$\text{Number of } \Gamma^1 = \frac{6}{6} = 1$$

$$\text{Number of } \Gamma^2 = \frac{8-2}{6} = 1$$

Correct answer: (c)

53. Light of wavelength 660 nm and power of 1 mW is incident on a semiconductor photodiode with an absorbing layer of thickness of $(\ln 4) \mu\text{m}$

(A) If the absorption coefficient at this wavelength is 10^4 cm^{-1} and if 1% power is lost on reflection at the surface, the power absorbed will be

- (a) 750 μW (b) 675 μW (c) 250 μW (d) 225 μW

(B) The generated photo-current for a quantum efficiency of unity will be

- (a) 360 μA (b) 400 μA (c) 133 μA (d) 120 μA

(A) Power of incident light = 1000 μW

Power of reflector light = 10 μW

Power of light entering the semiconductor = 990 μW

Amount existing in the semiconductor = $(990 \mu\text{W}) e^{-\mu x}$

where $\mu = 10^4 \text{ cm}^{-1}$, $x = (\ln 4) \mu\text{m} = (\ln 4) \times 10^{-4} \text{ cm}$

Therefore, amount exiting in the semiconductor = $(990 \mu\text{W}) e^{-\ln 4} = (990 \mu\text{W}) e^{(\ln 4)^{-1}} = \frac{1}{4} \times 990 \mu\text{W}$

Therefore, amount of power transmitted through semiconductor = $\left(990 - \frac{1}{4} \times 990\right) \mu\text{W} \approx 750 \mu\text{W}$

So, power absorbed in the semiconductor = $(1000 \mu\text{W} - 750 \mu\text{W}) = 250 \mu\text{W}$

Correct answer = 250 μW

[Note: If 1% light is not reflected only then the exact answer will be 750 μW .

Correct answer : (c)

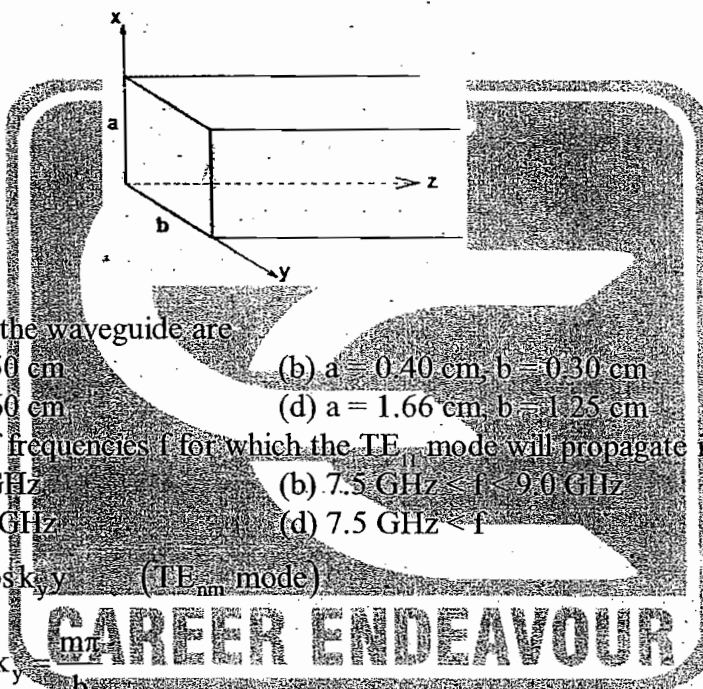
(B) $Q.E. = \frac{R \times 1246}{\lambda(\text{nm})}$ where R is responsivity and Q.E. is quantum efficiency.

$$\text{Also, } R = \frac{I_{op}(\text{A})}{P_{in}(\text{W})} \text{ and } Q.E. = 1 \Rightarrow 1 = \frac{I_{op} \times 1246}{P_{in}(\text{W}) \times \lambda(\text{nm})}$$

$$\Rightarrow I_{op} = \frac{250 \times 10^{-6} \times 660}{1246} = 133 \times 10^{-6} \text{ A} = 133 \mu\text{A}.$$

Correct answer: (c)

54. The magnetic field of the TE_{11} mode of a rectangular waveguide of dimensions $a \times b$ as shown in the figure is given by $H_z = H_0 \cos(0.3\pi x) \cos(0.4\pi y)$, where x and y are in cm.



(A) The dimensions of the waveguide are

(a) $a = 3.33 \text{ cm}, b = 2.50 \text{ cm}$

(b) $a = 0.40 \text{ cm}, b = 0.30 \text{ cm}$

(c) $a = 0.80 \text{ cm}, b = 0.60 \text{ cm}$

(d) $a = 1.66 \text{ cm}, b = 1.25 \text{ cm}$

(B) The entire range of frequencies f for which the TE_{11} mode will propagate is:

(a) $6.0 \text{ GHz} < f < 7.5 \text{ GHz}$

(b) $7.5 \text{ GHz} < f < 9.0 \text{ GHz}$

(c) $7.5 \text{ GHz} < f < 12.0 \text{ GHz}$

(d) $7.5 \text{ GHz} < f$

Soln. (A) $H_z = H_0 \cos k_x x \cos k_y y$ (TE_{nm} mode)

where, $k_x = \frac{n\pi}{a}$

$k_y = \frac{m\pi}{b}$

For, $n = m = 1$ $k_x = \frac{\pi}{a}$, $k_y = \frac{\pi}{b}$

From the question, $k_x = \frac{\pi}{a} = 0.3\pi \Rightarrow a = 3.33 \text{ cm}$

$k_y = \frac{\pi}{b} = 0.4\pi \Rightarrow b = \frac{10}{4} = 2.5 \text{ cm}$

Correct answer: (a)

(B) Cut-off frequency

$$f_{c,mn} = \frac{c}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, (m, n = 0, 1, \dots; m = n \neq 0) = \frac{c}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

wave propagation begins when the frequency slightly exceeds $f_{c,mn}$.

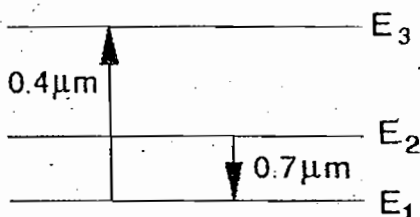
For TE_{11} , $m = 1, n = 1$.

$$(f_c)_{11} = \frac{c}{2} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \frac{3 \times 10^8}{2} \sqrt{\left(\frac{1}{0.0333}\right)^2 + \left(\frac{1}{0.0250}\right)^2} = 7.5 \times 10^9 \text{ Hz} = 7.5 \text{ GHz}$$

All the frequencies greater than cut-off frequency $(f_c)_{11}$ will be possible for TE_{11} .

Correct answer: (d).

55. 1.6×10^{19} Chromium ions per cubic centimeter. All the atoms excited by the $0.4 \mu\text{m}$ radiation decay rapidly to level E_2 which has a lifetime $\tau = 3\text{ms}$



(A) Assuming that there is no radiation of wavelength $0.7 \mu\text{m}$ present in the pumping cycle and that the pumping rate is R atoms per cm^3 , the population density in the level N_2 builds up as:

- (a) $N_2(t) = R\tau(e^{t/\tau} - 1)$ (b) $N_2(t) = R\tau(1 - e^{-t/\tau})$
 (c) $N_2(t) = \frac{Rt^2}{\tau}(1 - e^{-t/\tau})$ (d) $N_2(t) = Rt$

(B) The minimum pump power required (per cubic centimeter) to bring the system to transparency, i.e. zero gain, is

- (a) 1.52 kW (b) 2.64 kW (c) 0.76 kW (d) 1.32 kW

Soln. (A) $\frac{dN_2}{dt} = R - \lambda_2 N_2$

$$\Rightarrow \frac{dN_2}{(R/\lambda_2 - N_2)} = -\lambda_2 dt \Rightarrow \ln\left(\frac{R/\lambda_2 - N_2}{R/\lambda_2}\right) = -\lambda_2 t + c$$

Since at, $t = 0$ $N_2 = 0 \Rightarrow c = \ln\left(\frac{R}{\lambda_2}\right)$

$$\Rightarrow \ln\left(\frac{R/\lambda_2 - N_2}{R/\lambda_2}\right) = -\lambda_2 t \Rightarrow \frac{R/\lambda_2 - N_2}{R/\lambda_2} = e^{-\lambda_2 t} \Rightarrow N_2 = \frac{R}{\lambda_2}(1 - e^{-\lambda_2 t}), \left[\text{since, } \lambda_2 = \frac{1}{\tau}\right]$$

$$\Rightarrow N_2(t) = R\tau(1 - e^{-t/\tau})$$

Correct answer: (b)

(B) $(\text{Power})_{\text{Threshold}} = \frac{N(h\nu_p)}{2\tau_2} = \frac{N_2}{2\tau_2} \cdot \frac{hc}{\lambda_p}$

Putting, $N = 1.6 \times 10^{19}$, $\lambda_p = 0.4 \mu\text{m}$, $\tau_2 = 3\text{ms}$

We get, $(\text{Power})_{\text{Threshold}} = 1.32 \times 10^3 \text{ W} = 1.32 \text{ kW}$

Correct answer: (d)

56. A flux quantum (fluxoid) is approximately equal to 2×10^{-7} gauss-cm². A type II superconductor is placed in a small magnetic field, which is then slowly increased till the field starts penetrating the superconductor. The strength of the field at this point is $\frac{2}{\pi} \times 10^5$ gauss
- (A) The penetration depth of this superconductor is
 (a) 100 Å (b) 10 Å (c) 1000 Å (d) 314 Å
- (B) The applied field is further increased till superconductivity is completely destroyed. The strength of the field is now $\frac{8}{\pi} \times 10^5$ gauss. The correlation length of the superconductor is:
 (a) 20 Å (b) 200 Å (c) 628 Å (d) 2000 Å

Soln. (A) $\lambda^2 = \frac{\Phi_0}{\pi H_{c1}} \Rightarrow \lambda^2 = \frac{2 \times 10^{-7} \text{ (gauss-cm}^2\text{)}}{\pi \left(\frac{2}{\pi} \times 10^5 \right) \text{ gauss}} = 10^{-12} \text{ cm}^2 \Rightarrow H_{c1} = \frac{\Phi_0}{\pi \lambda^2}$ (λ = penetration depth)

$$\Rightarrow \lambda = 10^{-6} \text{ cm} = 100 \text{ Å}$$

Correct answer: (a)

(B) $H_{c2} = \frac{\Phi_0}{\pi \xi^2}$, (ξ = correlation length)

$$\Rightarrow \xi^2 = \frac{\Phi_0}{\pi H_{c2}} \Rightarrow \xi^2 = \frac{2 \times 10^{-7}}{\pi \left(\frac{8}{\pi} \times 10^5 \right) \text{ cm}^2} \Rightarrow \xi = 50 \text{ Å}$$

[Does not match with any option, however we can still eliminate the wrong options, ξ has different expression in different approximations]

For superconductor II

$$\frac{\lambda}{\xi} > \frac{1}{\sqrt{2}}$$

Only option (a) satisfies this.

Correct answer : (a)

57. A beam of pions (π^+) is incident on a proton target, giving rise to the process $\pi^+ p \rightarrow n + \pi^+ + \pi^+$
- (A) Assuming that the decay proceeds through strong interactions, the total isospin I and its third component I_3 for the decay products, are
 (a) $I = \frac{3}{2}, I_3 = \frac{3}{2}$ (b) $I = \frac{5}{2}, I_3 = \frac{5}{2}$ (c) $I = \frac{5}{2}, I_3 = \frac{3}{2}$ (d) $I = \frac{1}{2}, I_3 = -\frac{1}{2}$
- (B) Using isospin symmetry, the cross-section for the above process can be related to that of the process
 (a) $\pi^- n \rightarrow p \pi^- \pi^-$ (b) $\pi^- \bar{p} \rightarrow \bar{n} \pi^- \pi^-$ (c) $\pi^+ n \rightarrow p \pi^+ \pi^-$ (d) $\pi^+ \bar{p} \rightarrow n \pi^+ \pi^-$

Soln. (A) $\pi^+ p \rightarrow n + \pi^+ + \pi^+$

Decay products n, π^+, π^+

Decay product	Iso spin (I)	Third component (I_3)
n	$\frac{1}{2}$	$-\frac{1}{2}$
π^+	1	+1
π^+	1	+1
Total	$\frac{5}{2}$	$\frac{3}{2}$

$$I = 5/2, I_3 = 3/2.$$

Correct answer : (c)

Note: Reaction is forbidden $\left(\because I = \frac{3}{2}, I_3 = \frac{3}{2} \text{ for reactants} \right)$. For strong interaction but is allowed by EM interaction.

(B) In isospin symmetry, the particles are replaced by the corresponding antiparticles

Reaction of particles

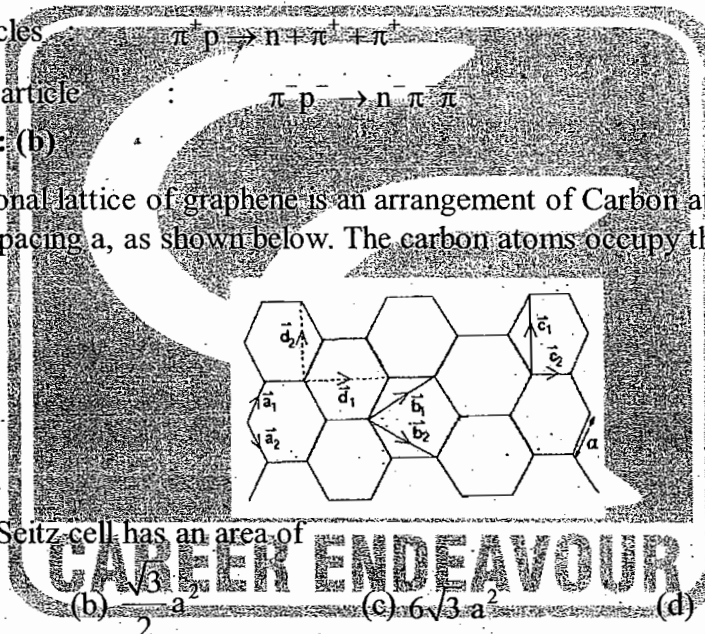
$$\pi^+ p \rightarrow n + \pi^+ + \pi^+$$

Reaction of antiparticle

$$\pi^- p \rightarrow n + \pi^- + \pi^-$$

Correct answer : (b)

58. The two dimensional lattice of graphene is an arrangement of Carbon atoms forming a honeycomb lattice of lattice spacing a , as shown below. The carbon atoms occupy the vertices.



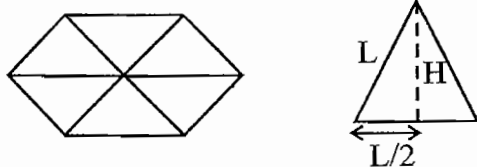
(A) The Wigner-Seitz cell has an area of

- (a) $2a^2$ (b) $\frac{\sqrt{3}}{2}a^2$ (c) $6\sqrt{3}a^2$ (d) $\frac{3\sqrt{3}}{2}a^2$

(B) The Bravais lattice for this array is a

- (a) Rectangular lattice with basis vectors $\bar{d}_1$ and $\bar{d}_2$
 (b) Rectangular lattice with basis vectors $\bar{c}_1$ and $\bar{c}_2$
 (c) Hexagonal lattice with basis vectors $\bar{a}_1$ and $\bar{a}_2$
 (d) Hexagonal lattice with basis vectors $\bar{b}_1$ and $\bar{b}_2$.

Soln. (A) The Wigner Seitz cell has the area of a hexagon. Split the hexagon into six equilateral triangles.



Use Pythagoras theorem

$$L^2 = \left(\frac{L}{2}\right)^2 + H^2 \Rightarrow H = \sqrt{L^2 - \frac{L^2}{4}} = \frac{\sqrt{3}}{2}L$$

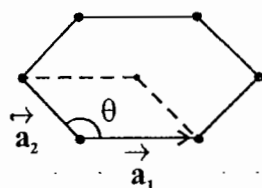
$$\text{Area of triangle} = \frac{1}{2}(\text{base}) \times (\text{height}) = \frac{1}{2}(L) \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$$

$$\text{Area of hexagon} = 6 \times \frac{\sqrt{3}}{4}L^2 = \frac{3\sqrt{3}}{2}L^2$$

Correct answer : (d)

(B) The hexagonal Bravais lattice is defined as

$$|\vec{a}_1| = |\vec{a}_2|, \quad \theta = 120^\circ$$



Correct answer: (c)

59. Consider the decay process $\tau^- \rightarrow \pi^- + \nu_\tau$ in the rest frame of the τ^- . The masses of τ^- , π^- and ν_τ are M_τ , M_π and zero respectively.

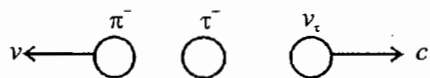
(A) The energy of π^- is

(a) $\frac{(M_\tau^2 - M_\pi^2)c^2}{2M_\tau}$ (b) $\frac{(M_\tau^2 + M_\pi^2)c^2}{2M_\tau}$ (c) $(M_\tau - M_\pi)c^2$ (d) $\sqrt{M_\tau M_\pi}c^2$

(B) The velocity of π^- is

(a) $\frac{(M_\tau^2 - M_\pi^2)c}{M_\tau^2 + M_\pi^2}$ (b) $\frac{(M_\tau^2 - M_\pi^2)c}{M_\tau^2 - M_\pi^2}$ (c) $\frac{M_\pi c}{M_\tau}$ (d) $\frac{M_\tau c}{M_\pi}$

Soln. $\tau^- \rightarrow \pi^- + \nu_\tau$



Let energy of π^- is E .

Therefore, energy of ν_τ is $M_\tau c^2 - E$

Therefore, momentum of ν_τ is $\frac{M_\tau c^2 - E}{c}$

Momentum of π^- is equal and opposite to that of ν_τ .

Using energy momentum relation for π^- $\left(E = \sqrt{p^2 c^2 + M_\pi^2 c^4}\right)$

We get, $E^2 = (M_\tau c^2 - E)^2 + M_\pi^2 c^4$

Or, $E^2 = M_\tau^2 c^4 - 2EM_\tau c^2 + E^2 + M_\pi^2 c^4$

$$\therefore \boxed{E = \frac{M_\tau^2 + M_\pi^2}{2M_\tau} c^2}$$

The correct answer is (b) (A)

Now, momentum of π^- is

$$p = \frac{M_\tau c^2 - E}{c} = \frac{M_\tau^2 - M_\pi^2}{2M_\tau} c$$

Velocity, $v = \frac{p}{E} c^2 = \frac{M_\tau^2 - M_\pi^2}{M_\tau^2 + M_\pi^2} c$

Correct answer is (a) (B)

60. A narrow beam of X-rays with wavelength $1.5 \text{ \AA}$ is reflected from an ionic crystal with an fcc lattice structure with a density of 3.32 g cm^{-3} . The molecular weight is 108 AMU ($1 \text{ AMU} = 1.66 \times 10^{-24} \text{ g}$).

(A) The lattice constant is:

- (a) $6.00 \text{ \AA}$ (b) $4.56 \text{ \AA}$ (c) $4.00 \text{ \AA}$ (d) $2.56 \text{ \AA}$

(B) The sine of the angle corresponding to (111) reflection is

- (a) $\sqrt{3}/4$ (b) $\sqrt{3}/8$ (c) $1/4$ (d) $1/8$

Soln. (A) Density $\rho = \frac{NM}{V} = \frac{NM}{a^3}$

$N = 4$ (Number of atoms in fcc lattice)

$M = 108 \times 1.66 \times 10^{-24} \text{ g}$, $\rho = 3.32 \text{ g/cm}^3$

$$\Rightarrow a^3 = \frac{4 \times 108 \times 1.66 \times 10^{-24}}{3.32} \text{ cm}^3 \Rightarrow a^3 = 216 \text{ \AA}^3 \Rightarrow a = 6 \text{ \AA}$$

Correct answer: (a)

(B) $2d_{hkl} \sin \theta = n\lambda$ (Bragg's law)

For $\boxed{n=1}$ $2d_{hkl} \sin \theta = \lambda$ where d_{hkl} is interplanar spacing.

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + \ell^2}} \Rightarrow d_{111} = \frac{a}{\sqrt{3}} \text{ where } a = 6 \text{ \AA}$$

Using Bragg's law, $\Rightarrow 2 \times \frac{6}{\sqrt{3}} \sin \theta = 1.5 \Rightarrow \sin \theta = \frac{1.5}{12} \times \sqrt{3} = \frac{\sqrt{3}}{8}$

Correct answer : (b)

61. If an electron is in the ground state of the hydrogen atom, the probability that its distance from the proton is more than one Bohr radius is approximately

(a) 0.68 (b) 0.48 (c) 0.28 (d) 0.91

Soln. $\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \Rightarrow |\psi_{100}(r)|^2 = \frac{1}{\pi a_0^3} e^{-2r/a_0}$

Probability for electron to lie outside the first Bohr radius is:

$$P = \int_{a_0}^{\infty} |\psi_{100}(r)|^2 4\pi r^2 dr = \int_{a_0}^{\infty} \frac{4\pi}{\pi a_0^3} r^2 e^{-2r/a_0} dr = \frac{4}{a_0^3} \int_{a_0}^{\infty} r^2 e^{-2r/a_0} dr = \frac{5}{e^2} = 0.68$$

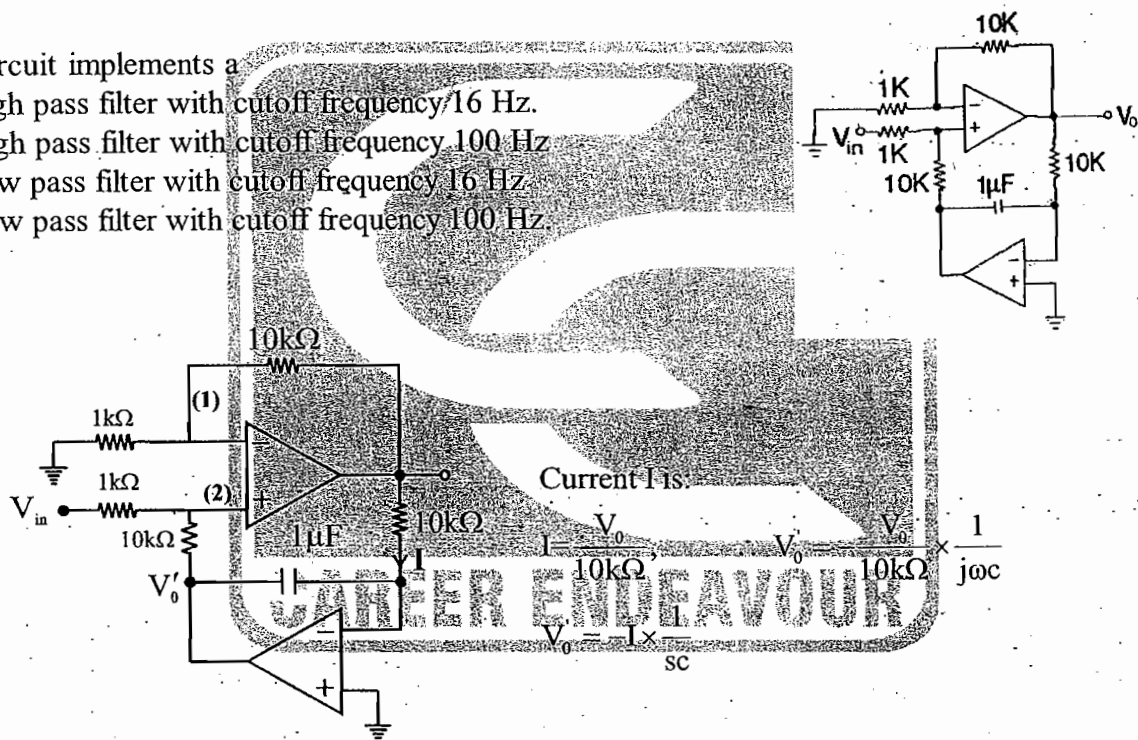
Correct answer: (a)

62. A time varying signal V_{in} is fed to an op-amp circuit with output signal V_o as shown in the figure below.

The circuit implements a

- (a) High pass filter with cutoff frequency 16 Hz.
 (b) High pass filter with cutoff frequency 100 Hz
 (c) Low pass filter with cutoff frequency 16 Hz
 (d) Low pass filter with cutoff frequency 100 Hz.

Soln.



Apply KCL at node (2)

$$\frac{V_1 - V_{in}}{1k\Omega} + \frac{V_1 - V_o}{10k\Omega} = 0$$

Apply KCL at node (1)

$$\frac{V_1 - 0}{1k\Omega} + \frac{V_1 - V_o}{10k\Omega} = 0$$

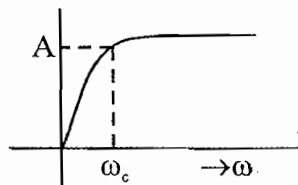
$$\Rightarrow 11V_1 = V_o \Rightarrow V_1 = \frac{V_o}{11}$$

$$\frac{V_o/11 - V_{in}}{1k\Omega} + \frac{V_o/11 + \frac{V_o}{10k\Omega}}{10k\Omega} = 0 \Rightarrow \frac{10V_o}{11} + \frac{V_o}{11} + \frac{V_o}{11k\Omega} = 10V_{in}$$

$$V_o(s) \left[1 + \frac{1}{j\omega c \times 10k} \right] = 10V_{in}(s) \Rightarrow \frac{V_o(s)}{V_{in}(s)} = \frac{10}{1 + \frac{1}{j\omega c \times 10k\Omega}}$$

$$\frac{V_o(j\omega)}{V_{in}(j\omega)} = \frac{10}{1 + \frac{1}{j\omega \times 10^{-6} \times 10^{-4}}}$$

as $\omega \rightarrow 0$	$A \rightarrow 0$
$\omega \rightarrow \infty$	$A \rightarrow 10$



Cut of frequency is $\left(\frac{10^4}{\omega_c}\right) = 1$

$$\omega_c = 10^2$$

$$f_c = \frac{100}{2\pi} \cong 16\text{Hz}$$

So, it high pass filter with cutoff frequency 16 Hz.

63. The Hamiltonian of a particle of unit mass moving in the xy -plane is given to be:

$$H = xp_x - yp_y - \frac{1}{2}x^2 + \frac{1}{2}y^2 \text{ in suitable units. The initial values are given to be } (x(0), y(0)) = (1, 1)$$

and $(p_x(0), p_y(0)) = \left(\frac{1}{2}, -\frac{1}{2}\right)$. During the motion, the curves traced out by the particles in the xy -plane and the $p_x p_y$ plane are

- (a) Both straight lines (b) A straight line and a hyperbola respectively
(c) A hyperbola and ellipse, respectively (d) Both hyperbolas

Soln. $H = xp_x - yp_y - \frac{x^2}{2} + \frac{y^2}{2}$

Curves in $x-y$ and p_x-p_y plane represent relation between x and y and p_x and p_y respectively.

Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p_x} = x \Rightarrow \dot{x} = x \Rightarrow \frac{dx}{dt} = x \Rightarrow \int \frac{dx}{x} = \int dt \Rightarrow x = e^t \quad \dots(i)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = -y \Rightarrow \dot{y} = -y \Rightarrow \frac{dy}{dt} = -y \Rightarrow \int \frac{dy}{y} = -\int dt \Rightarrow y = e^{-t} \quad \dots(ii)$$

Multiplying (i) and (ii) we get, $xy = 1$, this is equation of rectangular hyperbola in $x-y$ plane.

Now,

Hamilton's equations:

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -p_x + x \Rightarrow \frac{dp_x}{dt} + p_x = e^t \Rightarrow p_x e^t = \frac{1}{2} e^{2t} + c_1$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = p_y + y \Rightarrow \frac{dp_y}{dt} - p_y = y = e^{-t} \Rightarrow p_y e^{-t} = -\frac{1}{2} e^{-2t} + c_2$$

Using initial conditions. $(p_x(0), p_y(0)) = \frac{1}{2}, -\frac{1}{2}$

We get, $c_1 = 0, c_2 = 0$

$$\therefore p_x = \frac{1}{2}e^t \text{ and } p_y = -\frac{1}{2}e^{-t}$$

Multiplying p_x and p_y , we get

$$p_x \cdot p_y = -\frac{1}{4}$$

This is also equation of rectangular hyperbola.

Correct answer : (d)

64. Consider an ideal Bose gas in three dimensions with the energy-momentum relation $\epsilon \propto p^s$ with $s > 0$. The range of s for which this system may undergo a Bose-Einstein condensation at a non-zero temperature is:

(a) $1 < s < 3$ (b) $0 < s < 2$ (c) $0 < s < 3$ (d) $0 < s < \infty$

Soln. $\epsilon = ap^s \Rightarrow d\epsilon = asp^{s-1} dp$

Density of states = $D(\epsilon) d\epsilon = \frac{V}{h^3} 4\pi p^2 dp$

Now, $p = \left(\frac{\epsilon}{a}\right)^{1/s} \Rightarrow p^2 = \left(\frac{\epsilon}{a}\right)^{2/s}$ and $dp = \frac{1}{a^{1/s}} \cdot \frac{1}{s} \epsilon^{1/s-1} d\epsilon = \frac{1}{a^{1/s} s} \epsilon^{1/s-1} d\epsilon$

So, $D(\epsilon) d\epsilon = \frac{V}{h^3} 4\pi \left(\frac{\epsilon}{a}\right)^{2/s} \cdot \frac{1}{a^{1/s} s} \epsilon^{1/s-1} d\epsilon$; $D(\epsilon) d\epsilon = \frac{4\pi V}{h^3} \frac{1}{s a^{3/s}} \epsilon^{3/s-1} d\epsilon$

Number of particular N' that can be accommodated in all the excited states for Bose-Einstein condensation is

$$N' = \int_0^\infty \frac{D(\epsilon) d\epsilon}{e^{\beta\epsilon} - 1}, \quad (\text{chemical potential } \mu = 0)$$

$$= \frac{4\pi V}{h^3} \frac{1}{s a^{3/s}} \int_0^\infty \frac{\epsilon^{3/s-1} d\epsilon}{e^{\beta\epsilon} - 1}$$

For $s = 0$, integrated $\epsilon^{(3/s)}$ blows; $s = 1$, $\int_0^\infty \frac{\epsilon^2 d\epsilon}{(e^{\beta\epsilon} - 1)}$, integral blows.

$$s = 3, \int_0^\infty \frac{d\epsilon}{(e^{\beta\epsilon} - 1)}, \text{ integrated blows at } \epsilon = 0. \therefore 1 < s < 3$$

Correct answer: (a)

65. Two gravitating bodies A and B with masses m_A and m_B , respectively, are moving in circular orbit. Assume that $m_B \gg m_A$ and let the radius of the orbit of body A be R_A . If the body A is losing mass adiabatically, its orbital radius R_A is proportional to

(a) $1/m_A$ (b) $1/m_A^2$ (c) m_A (d) m_A^2

Soln. As the mass loss is adiabatic, **angular momentum is conserved**. [One can consider m_B to be the mass of the Sun while m_A to be the mass of earth, as $m_B \gg m_A$].

Now, $m_A R_A V_A = m_A R_B V_B = \text{constant} \quad \dots (1)$

R_B is the new radius.

Again, $m_A V_A^2 / R_A = GM_B m_A / R_A^2 \quad \dots (2)$

Thus, replacing V_A in (1) from (2) we obtain, $m_A^2 R_A = \text{constant}$.

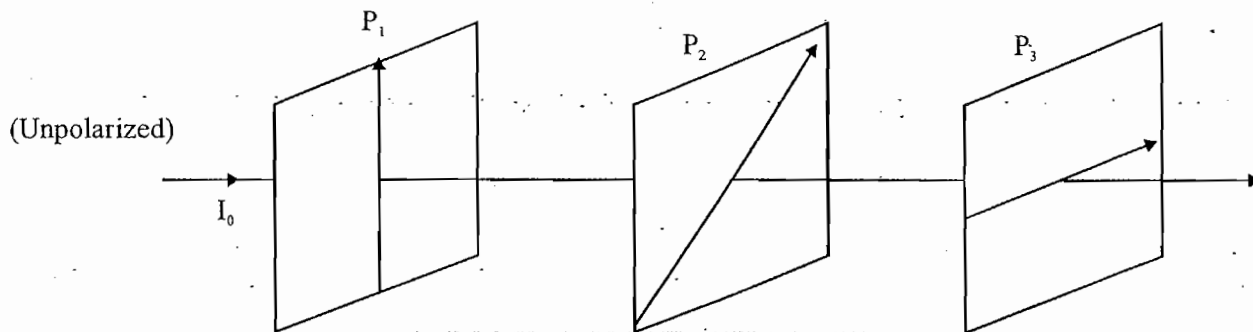
Correct answer: (b).



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PHYSICAL SCIENCES BOOKLET - [B]

Part B

21. Consider three polarizers P_1, P_2 and P_3 placed along an axis as shown in the Figure.



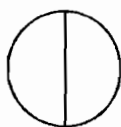
The pass axis of P_1 and P_3 are at right angles to each other while the pass axis of P_2 makes an angle θ with that of P_1 . A beam of unpolarized light of intensity I_0 is incident on P_1 as shown. The intensity of light emerging from P_3 is

- (a) 0 (b) $\frac{I_0}{2}$ (c) $\frac{I_0}{8} \sin^2 2\theta$ (d) $\frac{I_0}{4} \sin^2 2\theta$

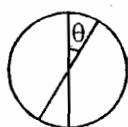
Soln. After passing P_1 the intensity $= I_0 / 2$ (Random)

After passing P_2 the intensity $= \left(\frac{I_0}{2}\right) \cos^2 \theta$ (Malus law)

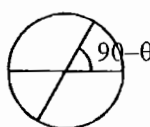
After passing P_3 the intensity $= \left(\frac{I_0}{2}\right) \cos^2 \theta \cos^2 (90^\circ - \theta)$
 $= \left(\frac{I_0}{2}\right) \cos^2 \theta \sin^2 \theta = \frac{I_0}{8} \sin^2 2\theta$



P_1



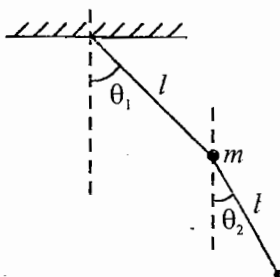
P_2



P_3

Correct answer is (c).

22. A double pendulum consists of two point masses m attached by massless strings of length l as shown in the figure:



The kinetic energy of the pendulum is :

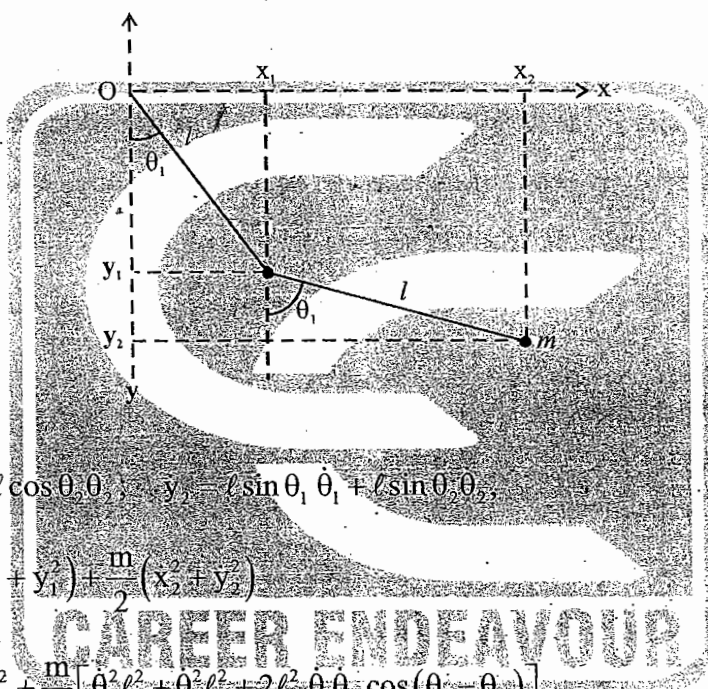
(a) $\frac{1}{2} m \ell^2 [\dot{\theta}_1^2 + \dot{\theta}_2^2]$ (b) $\frac{1}{2} m \ell^2 [2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]$

(c) $\frac{1}{2} m \ell^2 [\dot{\theta}_1^2 + 2\dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]$ (d) $\frac{1}{2} m \ell^2 [2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2)]$

Soln. $x_1 = \ell \sin \theta_1, \quad x_2 = \ell \sin \theta_1 + \ell \sin \theta_2$

$$y_1 = -\ell \cos \theta_1, \quad y_2 = -\ell \cos \theta_1 - \ell \cos \theta_2$$

$$\dot{x}_1 = \ell \cos \theta_1 \dot{\theta}_1, \quad \dot{y}_1 = \ell \sin \theta_1 \dot{\theta}_1$$

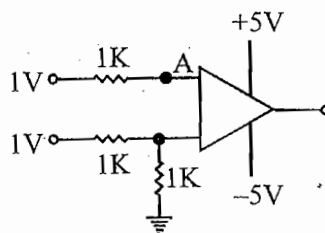


$$\dot{x}_2 = \ell \cos \theta_1 \dot{\theta}_1 + \ell \cos \theta_2 \dot{\theta}_2, \quad \dot{y}_2 = -\ell \sin \theta_1 \dot{\theta}_1 - \ell \sin \theta_2 \dot{\theta}_2$$

$$\begin{aligned} T &= \frac{m}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2} (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{m}{2} \dot{\theta}_1^2 \ell^2 + \frac{m}{2} [\dot{\theta}_1^2 \ell^2 + \dot{\theta}_2^2 \ell^2 + 2\dot{\theta}_1 \dot{\theta}_2 \ell^2 \cos(\theta_1 - \theta_2)] \\ &= \frac{m \ell^2}{2} [2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] \end{aligned}$$

Correct answer is (b).

23. In the operational amplifier circuit below, the voltage at point A is



- (a) 1.0 V (b) 0.5 V (c) 0 V (d) -5.0V

Soln. As the operational amplifier is open loop. So, concept of virtual ground is not applicable.

Correct answer is (a).

24. A point particle of mass m carrying an electric charge q is attached to a spring of stiffness constant k . A constant electric field E along the direction of the spring is switched on for a time interval T (where $T \ll \sqrt{m/k}$). Neglecting radiation loss, the amplitude of oscillation after the field is switched off is :

(a) qE/k (b) qET^2/m (c) $qE\sqrt{m}/Tk^{3/2}$ (d) $qET/\sqrt{mk}$

Soln. $m\ddot{y} = qE - ky \Rightarrow \ddot{y} + \omega^2 y = \frac{qE}{m}$, where, $\omega^2 = \frac{k}{m}$

Homogeneous solutions are $y_1 = \cos \omega t$, $y_2 = \sin \omega t$

Wronskian, $W = \begin{vmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{vmatrix} = 1$

$$y_p = -\cos \omega t \int (\sin \omega t) \frac{qE}{m} dt + \sin \omega t \int (\cos \omega t) \frac{qE}{m} dt$$

$$= \frac{\cos^2 \omega t}{\omega} \frac{qE}{m} + \frac{\sin^2 \omega t}{\omega} \frac{qE}{m}$$

$$\frac{dx}{dt} = \frac{qE}{m\omega} = \frac{qE}{m\sqrt{k/m}} = \frac{qE}{\sqrt{km}} \Rightarrow x_{\max} = \frac{qET}{\sqrt{km}}$$

Correct answer is (d).

25. A constant force F is applied to a relativistic particle of rest mass m . If the particle starts from rest at $t = 0$, its speed after a time t is

(a) Ft/m (b) $c \tanh\left(\frac{Ft}{mc}\right)$ (c) $c\left(1 - e^{-Ft/mc}\right)$ (d) $\frac{Fct}{\sqrt{F^2 t^2 + m^2 c^2}}$

Soln. $F = \frac{dp}{dt} \Rightarrow p = Ft$

$$E^2 = p^2 c^2 + m_0^2 c^4 \text{ and } E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \frac{m_0^2 c^4}{1 - v^2/c^2} = F^2 t^2 c^2 + m_0^2 c^4$$

$$\frac{m_0^2 c^2}{F^2 t^2 + m_0^2 c^2} = 1 - \frac{v^2}{c^2} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{m_0^2 c^2}{F^2 t^2 + m_0^2 c^2}$$

$$v = \frac{Ft}{\sqrt{F^2 t^2 + m_0^2 c^2}}$$

Correct answer is (d).

26. The potential of a diatomic molecule as a function of the distance r between the atoms is given by

$V(r) = -\frac{a}{r^6} + \frac{b}{r^{12}}$. The value of the potential at equilibrium separation between the atoms is :

- (a) $-4a^2/b$ (b) $-2a^2/b$ (c) $-a^2/2b$ (d) $-a^2/4b$

Soln. $V(r) = -\frac{a}{r^6} + \frac{b}{r^{12}} \Rightarrow \frac{\partial V}{\partial r} = \frac{6a}{r^7} - \frac{12b}{r^{13}} = 0$

At equilibrium $r = r_0$, $\frac{\partial V}{\partial r} = 0$

$$\Rightarrow 6a = \frac{12b}{r_0^6} \Rightarrow r_0^6 = \frac{2b}{a}$$

Therefore, at $r = r_0$, $V(r_0) = -\frac{a}{r_0^6} + \frac{b}{r_0^{12}} = \frac{-a}{\frac{2b}{a}} + \frac{b}{\left(\frac{2b}{a}\right)^2} = \frac{-a^2}{2b} + \frac{a^2}{4b} = \frac{-a^2}{4b}$

Correct answer is (d).

27. Four equal point charges are kept fixed at the four vertices of a square. How many neutral points (i.e., points where the electric field vanishes) will be found inside the square?

- (a) 3 (b) 4 (c) 5 (d) 7

Soln. One neutral point is at the centre of the square. By symmetry of the square, there are four more points, so in total there are 5 neutral points.

Correct answer is (c).

28. A static charge distribution gives rise to an electric field of the form $\vec{E} = \alpha(1 - e^{-r/R}) \frac{\hat{r}}{r^2}$, where α and R are positive constants. The charge contained within a sphere of radius R , centred at the origin is:

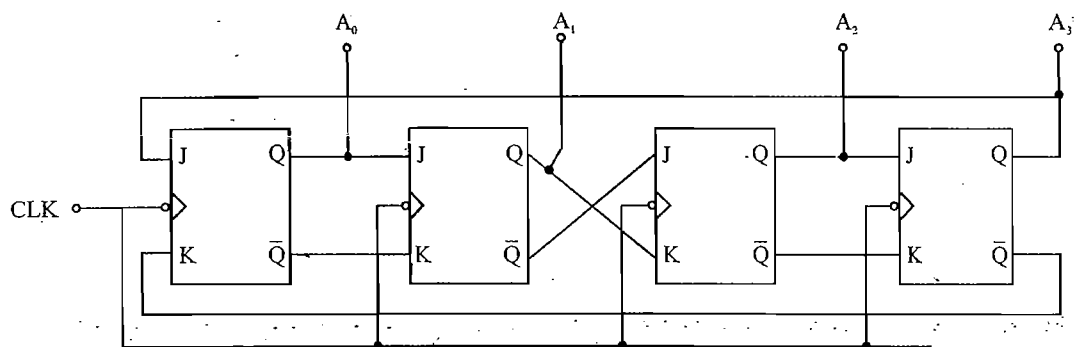
- (a) $\pi\alpha\epsilon_0 \frac{e}{R^2}$ (b) $\pi\alpha\epsilon_0 \frac{e^2}{R^2}$ (c) $4\pi\alpha\epsilon_0 \frac{R}{e}$ (d) $\pi\alpha\epsilon_0 \frac{R^2}{e}$

Soln. $E_r = \frac{\alpha}{r^2}(1 - e^{-r/R}) \Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 E_r) = \frac{\alpha}{Rr^2} e^{-r/R} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \frac{\alpha\epsilon_0}{R} \frac{e^{-r/R}}{r^2}$

$$\text{Total charge} = Q = \int_0^R 4\pi r^2 \rho dr = \frac{4\pi\alpha\epsilon_0}{R} \int_0^R e^{-r/R} dr = \frac{4\pi\alpha\epsilon_0}{R} R \left(1 - \frac{1}{e}\right) = 4\pi\alpha\epsilon_0 \left(1 - \frac{1}{e}\right)$$

No correct option is given, all options have incorrect dimension. $\alpha\epsilon_0$ has dimension of charge, which is correct in our calculation.

29. A counter consists of four flip-flops connected as shown in the figure.



If the counter is initialized as $A_0 A_1 A_2 A_3 = 0110$, the state after the next clock pulse is

- (a) 1000 (b) 0001 (c) 0011 (d) 1100

Soln. $Q_0 \quad Q_1 \quad Q_2 \quad Q_3 = 0110$

$$J_0 = Q_3 \quad J_1 = Q_0$$

$$K_0 = \bar{Q}_3 \quad K_1 = \bar{Q}_0$$

$$J_0 = 0 \quad J_1 = 0$$

$$K_0 = 1 \quad K_1 = 1$$

After one cycle

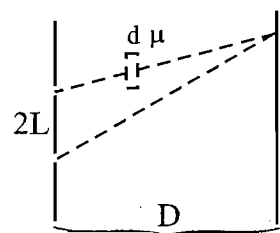
$$Q_0 = 0 \quad Q_1 = 0 \quad Q_2 = 0 \quad Q_3 = 1$$

Correct answer is (b).

30. In a Young's double slit interference experiment, the slits are at a distance $2L$ from each other and the screen is at a distance D from the slits. If a glass slab of refractive index μ and thickness d is placed in the path of one of the beams, the minimum value of d for the central fringe to be dark is

- (a) $\frac{\lambda D}{(\mu-1)\sqrt{D^2 + L^2}}$ (b) $\frac{\lambda D}{(\mu-1)L}$ (c) $\frac{\lambda}{(\mu-1)}$ (d) $\frac{\lambda}{2(\mu-1)}$

Soln.

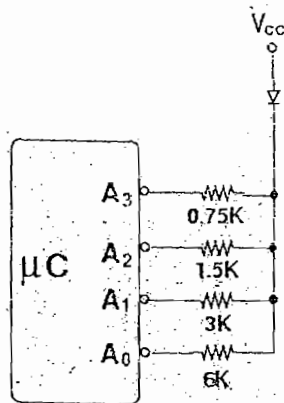


Extra path difference introduced $= (\mu-1)d$; here, $\frac{\lambda}{2}$ path difference needed to shift maxima to minima

$$\therefore (\mu-1)d = \frac{\lambda}{2} \Rightarrow d = \frac{\lambda}{2(\mu-1)}$$

Correct answer is (d).

31. The pins 0, 1, 2, and 3 of part A of a microcontroller are connected with resistors to drive an LED at various intensities as shown in the figure.



For $V_{cc} = 4.2V$ and a voltage drop of $1.2V$ across the LED, the range (maximum current) and resolution (Step size) of the drive current are, respectively.

- (a) 4.0 mA and 1.0 mA (b) 15.0 mA and 1.0 mA
(c) 7.5 mA and 0.5 mA (d) 4.0 mA and 0.5 mA

Soln. Potential at point P will be $4.2 - 1.2 = 3.0V$. For maximum current value of resistance should be minimum.

So, all 4 resistors will be in parallel

$$\frac{1}{R_{eq}} = \frac{1}{0.75} + \frac{1}{1.5} + \frac{1}{3} + \frac{1}{6} \Rightarrow \frac{1}{R_{eq}} = \frac{8+4+2+1}{6} \Rightarrow \frac{1}{R_{eq}} = \frac{6}{15}$$

$$\Rightarrow I_{max} = \frac{3}{6} \times 15 = 7.5\text{ mA}$$

For minimum current maximum resistance will be taken, $I_{min} = \frac{3}{6} = 0.5\text{ mA}$

Correct answer is (c).

32. An unbiased dice is thrown three times successively. The probability that the numbers of dots on the uppermost surface add up to 16 is

- (a) $\frac{1}{16}$ (b) $\frac{1}{36}$ (c) $\frac{1}{108}$ (d) $\frac{1}{216}$

Soln. The combination that add upto 16 are $(6, 5, 5), (5, 5, 6), (5, 6, 5), (6, 6, 4), (6, 4, 6), (4, 6, 6)$

Number of successful events = 6; Sample space = $6^3 = 216$

$$\text{Probability} = \frac{\text{Number of successful events}}{\text{Sample space}} = \frac{6}{6^3} = \frac{1}{36}$$

Correct answer is (b).

33. The generating function $F(x, t) \equiv \sum_{n=0}^{\infty} P_n(x) t^n$ for the Legendre polynomials $P_n(x)$ is

$$F(x, t) = (1 - 2xt + t^2)^{-1/2}. \text{ The value of } P_3(-1) \text{ is:}$$

- (a) 5/2 (b) 3/2 (c) +1 (d) -1

Soln. $F(x, t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$

Putting $x = -1$, we get

$$\Rightarrow (1 + t)^{-1} = \sum_{n=0}^{\infty} P_n(-1) t^n$$

$$\Rightarrow 1 - t + t^2 - t^3 + \dots = \sum_{n=0}^{\infty} P_n(-1) t^n$$

Comparing coefficient of t^3 on both side, $P_3(-1) = -1$.

Correct answer is (d).

34. Given that the ground state energy of the hydrogen atom is 13.6 eV, the ground state energy of positronium (which is a bound state of an electron and a positron) is

- (a) +6.8 eV (b) -6.8 eV (c) -13.6 eV (d) -27.2 eV

Soln. For H atom, $E_n = -R_M \frac{hc}{n^2} = \frac{13.6}{n^2} \text{ eV}$

where R_M = Rydberg constant which is proportional to reduced mass μ of the system.

For positronium, $\mu = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2}$

Energy of positronium atom, $E_n = -\frac{R_M}{2} \frac{hc}{n^2} = -\frac{6.8}{n^2} \text{ eV}$

Ground state energy = -6.8 eV

Correct answer is (b).

35. Two particles of identical mass move in circular orbits under a central potential $V(r) = \frac{1}{2} kr^2$. Let

ℓ_1 and ℓ_2 be the angular momenta and r_1, r_2 be the radii of the orbits respectively. If $\ell_1 / \ell_2 = 2$, the value of r_1 / r_2 is:

- (a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 2 (d) 1/2

Soln. $F(r) = \frac{\partial V}{\partial r} = \frac{\partial}{\partial r} \left(\frac{1}{2} kr^2 \right) = kr \Rightarrow \frac{mv^2}{r} = kr$

Since m and k are constant, $\frac{v^2}{r^2} = \text{constant} = \frac{v}{r} = \text{constant}$

$$\Rightarrow \ell = mvr \Rightarrow v = \frac{\ell}{mr} \Rightarrow \frac{\ell}{mr^2} = \text{constant}$$

$$\Rightarrow \frac{\ell}{r^2} = \text{constant} \Rightarrow \frac{r_1}{r_2} = \sqrt{\frac{\ell_1}{\ell_2}} = \sqrt{2}$$

Correct answer is (a).

36. The equation of the plane that is tangent to the surface $xyz = 8$ at the point $(1, 2, 4)$ is

(a) $x + 2y + 4z = 12$

(b) $4x + 2y + z = 12$

(c) $x + 4y + 2z = 12$

(d) $x + y + z = 7$

Soln. Consider a point $Q(x, y, z)$ on the tangent plane to the surface $xyz = 8$ at point $P(x, 1, 2, 4)$.

Therefore, $\overline{QP}$ lies on tangent plane.

So, $\overline{QP} \cdot \nabla \phi|_{(1,2,4)} = 0$ [Where $\nabla \phi$ is normal to the surface at point P]

$$\Rightarrow [(1-x)\hat{i} + (2-y)\hat{j} + (4-z)\hat{k}] \cdot (8\hat{i} + 4\hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow 4x + 2y + z = 12$$

Correct answer is (b).

37. The energy of the first excited quantum state of a particle in the two-dimensional potential

$$V(x, y) = \frac{1}{2}m\omega^2(x^2 + 4y^2) \text{ is}$$

(a) $2\hbar\omega$

(b) $3\hbar\omega$

(c) $\frac{3}{2}\hbar\omega$

(d) $\frac{5}{2}\hbar\omega$

Soln. $V(x, y) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m(2\omega)^2y^2$

$$E_n = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar \times 2\omega$$

For ground state $n_x = 0, n_y = 0; E_0 = \frac{3}{2}\hbar\omega$

For first excited state, $n_x = 1, n_y = 0 \Rightarrow E_1 = \frac{3}{2}\hbar\omega + \frac{1}{2}\hbar \times 2\omega = \frac{5}{2}\hbar\omega$

Correct answer is (d)

38. The internal energy E of a system is given by $E = \frac{bS^3}{VN}$, where b is a constant and other symbols have their usual meaning. The temperature of this system is equal to

(a) $\frac{bS^2}{VN}$ (b) $\frac{3bS^2}{VN}$ (c) $\frac{bS^3}{V^3N}$ (d) $\left(\frac{S}{N}\right)^2$

Soln. $E = \frac{bS^3}{VN}$

$$\Rightarrow dQ = TdS = dE + PdV - \mu dN$$

$$\Rightarrow T = \left(\frac{\partial E}{\partial S}\right)_{N,V} = \frac{3bS^2}{NV}$$

Correct answer is (b)

39. Consider a particle in a one dimensional potential that satisfies $V(x) = V(-x)$. Let $|\psi_0\rangle$ and $|\psi_1\rangle$ denote the ground and the first excited states, respectively, and let $|\psi\rangle = \alpha_0|\psi_0\rangle + \alpha_1|\psi_1\rangle$ be a normalized state with α_0 and α_1 being real constants. The expectation value $\langle x \rangle$ of the position operator x in the state $|\psi\rangle$ is given by

(a) $\alpha_0^2\langle\psi_0|x|\psi_0\rangle + \alpha_1^2\langle\psi_1|x|\psi_1\rangle$ (b) $\alpha_0\alpha_1[\langle\psi_0|x|\psi_1\rangle + \langle\psi_1|x|\psi_0\rangle]$
(c) $\alpha_0^2 + \alpha_1^2$ (d) $2\alpha_0\alpha_1$

Soln. For a symmetric potential, $\psi_n(x)$ has definite parity.

$$\langle x \rangle = \frac{\langle \psi | x | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$= \langle \psi_0 | \alpha_0 + \langle \psi_1 | \alpha_1 \rangle x \alpha_0 | \psi_0 \rangle + \alpha_1 | \psi_1 \rangle \rangle$$

$$= \alpha_0^2 \langle \psi_0 | x | \psi_0 \rangle + \alpha_1^2 \langle \psi_1 | x | \psi_1 \rangle + \alpha_0 \alpha_1 \langle \psi_0 | x | \psi_1 \rangle + \alpha_1 \alpha_0 \langle \psi_1 | x | \psi_0 \rangle$$

$\langle \psi_0 | x | \psi_0 \rangle = 0$ as 'x' is odd operator and ψ_n has definite parity. (Integral of even $\times$ odd function is zero) and similarly $\langle \psi_1 | x | \psi_1 \rangle = 0$.

$$\text{So, } \langle x \rangle = \alpha_0 \alpha_1 [\langle \psi_0 | x | \psi_1 \rangle + \langle \psi_1 | x | \psi_0 \rangle]$$

Correct answer is (b)

40. A 3×3 matrix M has $\text{Tr}[M] = 6$, $\text{Tr}[M^2] = 26$ and $\text{Tr}[M^3] = 90$. Which of the following can be a possible set of eigenvalues of M ?

(a) $\{1, 1, 4\}$ (b) $\{-1, 0, 7\}$ (c) $\{-1, 3, 4\}$ (d) $\{2, 2, 2\}$

Soln. Since, sum of eigenvalues of the matrix = Trace of the matrix.

$$\lambda_1 + \lambda_2 + \lambda_3 = 6;$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 26;$$

$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 90$$

It is trivial to check that only $\{-1, 3, 4\}$ satisfies these three equations.

Correct answer is (c).

41. The perturbation $H' = bx^4$, where 'b' is a constant, is added to the one dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Which of the following denotes the correction to the ground state energy to first order in b?

[Hint : The normalized ground state wave function of the one dimensional harmonic oscillator

potential is $\psi_0 = \left(\frac{m\omega}{h\pi}\right)^{1/4} e^{-\frac{m\omega x^2}{2h}}$. You may use the following integral $\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = a^{-n-\frac{1}{2}} \Gamma\left(n + \frac{1}{2}\right)$

(a) $\frac{3b\hbar^2}{4m^2\omega^2}$

(b) $\frac{3b\hbar^2}{2m^2\omega^2}$

(c) $\frac{3b\hbar^2}{2\pi m^2\omega^2}$

(d) $\frac{15b\hbar^2}{4m^2\omega^2}$

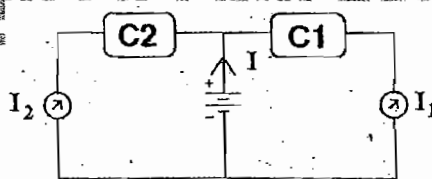
Soln.
$$I = \int_{-\infty}^{\infty} \psi_0^* H' \psi_0 dx = b \int_{-\infty}^{\infty} |\psi_0|^2 x^4 dx \Rightarrow \int_{-\infty}^{\infty} |\psi_0|^2 x^4 dx = \left(\frac{m\omega}{h\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{h}} x^4 dx$$

Comparing with the given standard integral $\alpha = \frac{m\omega}{h}$ and $n = 2$

$$I = b \left(\frac{m\omega}{h\pi}\right)^{1/2} \left(\frac{m\omega}{h}\right)^{-2} \Gamma\left(2 + \frac{1}{2}\right) \text{ since } \Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{3}{4} \sqrt{\pi} = \frac{3}{4} \frac{b\hbar^2}{m^2\omega^2}$$

Correct answer is (a).

42. A battery powers two circuits, C_1 and C_2 , as shown in the figure.



The total current I drawn from the battery is estimated by measuring the currents I_1 and I_2 through the individual circuits. If I_1 and I_2 are both 200 mA and if the errors in their measurement are 3 mA and 4 mA respectively, the error in the estimate of I is:

(a) 7.0 mA

(b) 7.5 mA

(c) 5.0 mA

(d) 10.5 mA

Soln. $P = P_1 + P_2$ (Powers)

$$\Rightarrow I^2 R = I_1^2 R_1 + I_2^2 R_2$$

$$\Rightarrow I^2 = I_1^2 + I_2^2 \quad (R = R_1 + R_2, \text{ since } I_1 = I_2)$$

$$\Rightarrow \Delta I = \sqrt{(\Delta I_1)^2 + (\Delta I_2)^2} = \sqrt{3^2 + 4^2} = 5.0 \text{ mA}$$

Correct answer is (c).

43. Consider a Maxwellian distribution of the velocity of the molecules of an ideal gas. Let V_{mp} and V_{rms} denote the most probable velocity and the root mean square velocity, respectively. The magnitude of the ratio V_{mp}/V_{rms} is:

(a) 1 (b) $2/3$ (c) $\sqrt{2/3}$ (d) $3/2$

Soln. $f(v) \propto v^2 \exp\left(-\frac{mv^2}{2kT}\right) \Rightarrow \frac{df}{dv} = 0 \Rightarrow v_{mp} = \sqrt{\frac{2kT}{m}}$

$$v_{rms} = \left[\int_0^\infty v^2 f(v) dv \right]^{1/2} = \sqrt{\frac{3kT}{m}} \quad \therefore \frac{v_{mp}}{v_{rms}} = \sqrt{\frac{2}{3}}$$

Correct answer is (c).

44. If the number density of a free electron gas in three dimensions is increased eight times, its Fermi temperature will

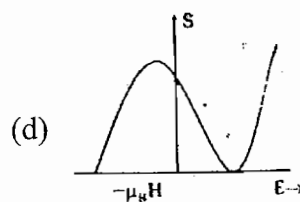
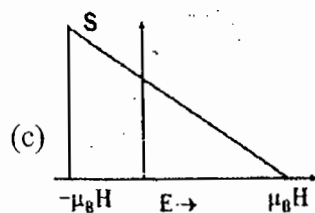
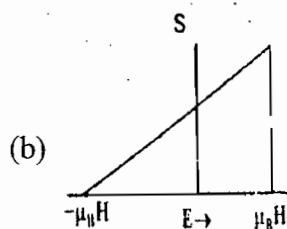
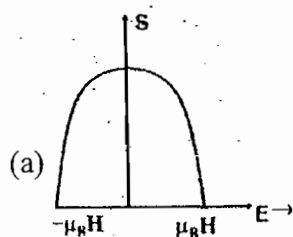
(a) increase by a factor of 4 (b) decrease by a factor of 4
(c) increase by a factor of 8 (d) decrease by a factor of 8

Soln. $E_F \propto n^{2/3}$ in 3-D

$$\frac{E_{F_2}}{E_{F_1}} = \left(\frac{n_2}{n_1}\right)^{2/3} = \left(\frac{8n_1}{n_1}\right)^{2/3} = (2^3)^{2/3} = 4 \Rightarrow E_{F_2} = 4 E_{F_1}$$

Correct answer is (a).

45. A system of N noninteracting spin $\frac{1}{2}$ particles is placed in an external magnetic field H . The behaviour of the entropy of the system as a function of energy is given by



Soln. There are N non-interacting particles.

Magnetic moment of each particle = μ

Energy of particle when aligned parallel with magnetic field $\mu = +\epsilon$

Energy of particle when aligned antiparallel with magnetic field $\mu = -\epsilon$

where $|\epsilon| = \mu H$

Total energy E of the isolated system is $E = n\epsilon = N_+ \epsilon - N_- \epsilon$, $n = N_+ - N_-$

$$N = N_+ + N_-, \quad N_- = \frac{1}{2}(N - n), \quad N_+ = \frac{1}{2}(N + n)$$

Number of Microstates, $\Omega = \frac{N!}{N_-! N_+!}$

$$\ln \Omega = \ln N! - \ln N_-! - \ln N_+!$$

$$= (N \ln N - N) - (N_- \ln N_- - N_-) - (N_+ \ln N_+ - N_+)$$

(We have used Stirling approximation)

$$= -[N_- \ln(N_-/N) + N_+ \ln(N_+/N)]$$

$$= -\left[\frac{1}{2}N \left(1 - \frac{n}{N}\right) \ln \frac{1}{2} \left(1 - \frac{n}{N}\right) + \frac{1}{2}N \left(1 + \frac{n}{N}\right) \ln \frac{1}{2} \left(1 + \frac{n}{N}\right) \right]$$

Using $\ln \left(1 \pm \frac{n}{N}\right) = \pm \frac{n}{N} - \frac{n^2}{2N^2} \pm \dots$

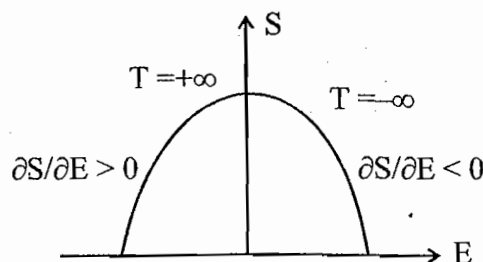
$$\ln \Omega = -N \ln \frac{1}{2} - \frac{n^2}{2N} \Rightarrow \ln \Omega = -N \ln \frac{1}{2} - \frac{(E/\epsilon)^2}{2N} = N \left[\ln 2 - \frac{(E/\epsilon)^2}{2N^2} \right]$$

Entropy $S = k \ln \Omega$

$$S = Nk \left[\ln 2 - \frac{(E/\epsilon)^2}{2N^2} \right] = Nk \left[\ln 2 - \frac{1}{2} \left(\frac{E}{N\epsilon} \right)^2 \right] = Nk \left[\ln 2 - \frac{1}{2} \left(\frac{E}{N\mu H} \right)^2 \right]$$

where $E = 0$, $S = Nk \ln 2$

The equation of S vs E represents the following curve.



Correct answer is (a).

Part-C

46. A particle of mass 'm' moves inside a bowl. If the surface of the bowl is given by the equation

$z = \frac{1}{2}a(x^2 + y^2)$, where a is a constant, the Lagrangian of the particle is:

- (a) $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 - gar^2)$ (b) $\frac{1}{2}m[(1 + a^2r^2)\dot{r}^2 + r^2\dot{\phi}^2]$
(c) $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\phi\dot{\phi}^2 - gar^2)$ (d) $\frac{1}{2}m[(1 + a^2r^2)\dot{r}^2 + r^2\dot{\phi}^2 - gar^2]$

Soln. $z = \frac{1}{2}a(x^2 + y^2) = \frac{1}{2}ar^2 \Rightarrow \dot{z} = ar\dot{r}$

$$\text{K.E.} = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + a^2r^2\dot{r}^2)$$

$$\text{P.E.} = V = mgz = \frac{1}{2}mgar^2$$

$$\therefore L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + a^2r^2\dot{r}^2 - gar^2) = \frac{1}{2}m[(1 + a^2r^2)\dot{r}^2 + r^2\dot{\phi}^2 - gar^2]$$

Correct answer is (d).

47. An electron of energy 27 GeV collides with a proton of energy 820 GeV. The heaviest particle which can be produced in this collision has mass close to

- (a) 300 GeV (b) 821 GeV (c) 850 GeV (d) 1127 GeV

Soln. Electron-proton interaction is mediated by Z- particles which has mass of 91 GeV. The total energy available is $820 + 27 = 847$ GeV, from where the Z- particle carries 91 GeV. The remaining 756 GeV is available to produce a jet of hadrons. Only option (a) is correct because $300 \text{ GeV} < 756 \text{ GeV}$.

Correct answer is (a).

48. Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of the differential equation

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + f(t)x = 0$, and let $w(t) = x_1(t)\frac{dx_2(t)}{dt} - x_2(t)\frac{dx_1(t)}{dt}$. If $w(0) = 1$, then $w(1)$ is given by

- (a) 1 (b) e^2 (c) $1/e$ (d) $1/e^2$

Soln. Given, differential equation is of form : $\frac{d^2x}{dt^2} + P(t)\frac{dx}{dt} + Q(t)x = 0$

By Abel's theorem, $W(t) = W(t_0) e^{-\int_{t_0}^t P(t) dt}$

$$\Rightarrow W(1) = W(0) e^{-\int_0^1 2dt} \Rightarrow W(1) = \frac{1}{e^2}$$

Correct answer is (d).

49. Assume that the free energy of a magnetic system has an expansion in the order parameter M of the form $F(M, T) = a(T - T_c)M^2 + bM^4 + cM^6$, with a , b and $c > 0$. As the temperature is lowered below T_c , the system undergoes a phase transition. The behaviour of the order parameter just below the transition, where $(T - T_c)$ is very small, is best described by

(a) $M \propto (T_c - T)^{-1/2}$ (b) $M \propto (T_c - T)^{1/2}$ (c) $M \propto (T_c - T)$ (d) $M \propto (T_c - T)^3$

Soln. Near the transition point, we can consider only the first two term $F = a(T - T_c)M^2 + bM^4$

$$\Rightarrow \frac{\partial F}{\partial M} = 2a(T - T_c)M + 4bM^3 = 0$$

$$\Rightarrow 2a(T - T_c) + 4bM^2 = 0$$

For this equation to be satisfied, $M^2 \sim (T - T_c)$ $M \sim (T - T_c)^{1/2}$

Correct answer is (b).

50. A planet of mass m moves in the inverse square central force field of the Sun of mass M . If the semi-major and semi-minor axes of the orbit are a and b , respectively, the total energy of the planet is :

(a) $-\frac{GMm}{a+b}$ (b) $-GMm\left(\frac{1}{a} + \frac{1}{b}\right)$ (c) $-\frac{GMm}{a}\left(\frac{1}{b} - \frac{1}{a}\right)$ (d) $-GMm\frac{(a-b)}{(a+b)^2}$

Soln. None of the option is correct. Correct answer is $-\frac{GMm}{2a}$.

51. Let $|0\rangle$ and $|1\rangle$ denote the normalized eigenstates corresponding to the ground and first excited state of a one dimensional harmonic oscillator. The uncertainty Δp in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, is:

(a) $\Delta p = \sqrt{\hbar m \omega} / 2$ (b) $\Delta p = \sqrt{\hbar m \omega} / 2$ (c) $\Delta p = \sqrt{\hbar m \omega}$ (d) $\Delta p = \sqrt{2 \hbar m \omega}$

Soln. $\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle = \frac{1}{2} [\langle 0 | \hat{p} | 0 \rangle + \langle 1 | \hat{p} | 1 \rangle + \langle 0 | \hat{p} | 1 \rangle + \langle 1 | \hat{p} | 0 \rangle]$

$$= -i \sqrt{\frac{\hbar m \omega}{2}} \times \frac{1}{2} [\langle 0 | (\hat{a} - \hat{a}^\dagger) | 0 \rangle + \langle 1 | (\hat{a} - \hat{a}^\dagger) | 1 \rangle + \langle 0 | (\hat{a} - \hat{a}^\dagger) | 1 \rangle + \langle 1 | (\hat{a} - \hat{a}^\dagger) | 0 \rangle]$$

$$= -\frac{i}{2} \sqrt{\frac{\hbar m \omega}{2}} [0 + 0 + \langle 0|1|0 \rangle - \langle 0|\sqrt{2}|2 \rangle + \langle 1|0|0 \rangle - \langle 1|1|1 \rangle] = 0$$

$$\hat{p}^2 = \left(-i \sqrt{\frac{\hbar m \omega}{2}} \right)^2 [\hat{a}^2 + \hat{a}^{\dagger 2} - \hat{a} \hat{a}^{\dagger} - \hat{a}^{\dagger} \hat{a}] \quad \left[\text{Since, } [\hat{a}, \hat{a}^{\dagger}] = 1 \Rightarrow \hat{a} \hat{a}^{\dagger} - \hat{a}^{\dagger} \hat{a} = 1 \right]$$

$$= -\frac{\hbar m \omega}{2} [\hat{a}^2 + \hat{a}^{\dagger 2} - 2\hat{a}^{\dagger} \hat{a} - 1]$$

$$\langle \hat{p}^2 \rangle = -\frac{\hbar m \omega}{2} \times \frac{1}{2} [\langle 0|\hat{a}^2|0 \rangle + \langle 1|\hat{a}^2|1 \rangle + \langle 0|\hat{a}^2|1 \rangle + \langle 1|\hat{a}^2|0 \rangle + \langle 0|\hat{a}^{\dagger 2}|0 \rangle + \langle 1|\hat{a}^{\dagger 2}|1 \rangle + \langle 0|\hat{a}^{\dagger 2}|1 \rangle + \langle 1|\hat{a}^{\dagger 2}|0 \rangle \\ - 2(\langle 0|\hat{a}^{\dagger} \hat{a}|0 \rangle + \langle 1|\hat{a}^{\dagger} \hat{a}|1 \rangle + \langle 0|\hat{a}^{\dagger} \hat{a}|1 \rangle + \langle 1|\hat{a}^{\dagger} \hat{a}|0 \rangle - (\langle 0|0 \rangle + \langle 1|1 \rangle) + \langle 0|1 \rangle + \langle 1|0 \rangle]$$

$$= -\frac{\hbar m \omega}{4} [-2(0+1) - 1(1+1)] = \hbar m \omega, \quad \Delta P = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} = \sqrt{\hbar m \omega}$$

Correct answer is (c).

52. A laser operating at 500 nm is used to excite a molecule. If the Stokes line is observed at 770 cm^{-1} , the approximate positions of the Stokes and the anti-Stokes lines are

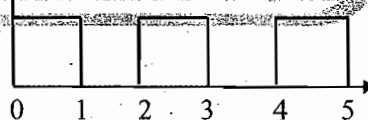
- (a) 481.5 nm and 520 nm (b) 481.5 nm and 500 nm
(c) 500 nm and 520 nm (d) 500 nm and 600 nm

Soln. Stokes and anti-Stokes lines never occur at the Laser frequency which is 500 nm. Only option (a) does not have this number.

Correct answer is (a).

53. The graph of the function

$$f(x) = \begin{cases} 1 & \text{for } 2n \leq x \leq 2n+1 \\ 0 & \text{for } 2n+1 \leq x \leq 2n+2 \end{cases}$$



(Where $n = 0, 1, 2, \dots$) is shown below.

Its Laplace transform $\tilde{f}(s)$ is

- (a) $\frac{1+e^{-s}}{s}$ (b) $\frac{1-e^{-s}}{s}$ (c) $\frac{1}{s(1+e^{-s})}$ (d) $\frac{1}{s(1-e^{-s})}$

Soln. Given function $f(x)$ is a periodic function of period $T = 2$

$$f(s) = \frac{1}{1-e^{-2s}} \int_0^T e^{-sx} f(x) dx = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-sx} \cdot 1 dx \right] \\ = \frac{1}{(1-e^{-2s})} \left(\frac{1-e^{-s}}{s} \right) = \frac{1}{s(1+e^{-s})}$$

Correct answer is (c).

54. The energy levels of electrons of mass 'm' and charge 'e' confined in an area A in the xy-plane with a uniform magnetic field B applied in the z-direction are given by $E_n = \left(n + \frac{1}{2}\right) \frac{eB\hbar}{mc}$, $n = 0, 1, 2, \dots$

The degeneracy of each level is $\frac{eBA}{\hbar c}$. The lowest level is completely filled the other are empty. The

fermi energy $\frac{\hbar^2 N}{2\pi mA}$, where N is the total number of electrons, is:

- (a) coincident with the $n = 0$ level (b) coincident with the $n = 1$ level
(c) midway between the $n = 0$ and the $n = 1$ levels
(d) midway between the $n = 1$ and the $n = 2$ levels.

Soln. Density of state $D(p) dp = \frac{A 2\pi p dp}{h^2}$

$$\epsilon = \frac{p^2}{2m},$$

$$D(\epsilon) d\epsilon = \frac{A 2\pi m d\epsilon}{h^2}$$

$$\text{Therefore, } N = \int_0^{\epsilon_F} A \frac{2\pi m}{h^2} d\epsilon = A \frac{2\pi m}{h^2} \epsilon_F \Rightarrow \epsilon_F = \frac{\hbar^2 N}{2\pi mA}$$

{But in the question Fermi energy is given as $\frac{\hbar^2 N}{2\pi mA}$ which is incorrect. This question is incorrect}.

Number of zero field states in an interval $\hbar\omega_c$ is

$$N = D(\epsilon) \hbar\omega_c = \left(\frac{A m}{\pi \hbar^2}\right) \hbar \frac{eB}{mc} \Rightarrow N = \frac{2e}{\hbar c} BA \Rightarrow E_n = \left(n + \frac{1}{2}\right) \frac{eB\hbar}{mc}$$

$$\epsilon_F = \frac{eB\hbar}{\pi mc} = \left(n + \frac{1}{2}\right) \frac{eB\hbar}{mc} \Rightarrow n = 1.5$$

The level $n = 1.5$ is between 1 and 2.

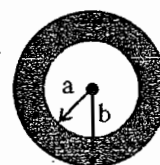
Correct answer is (d).

55. An annulus of mass M made of a material of uniform density has inner and outer radii 'a' and 'b' respectively. Its principal moment of inertia along the axis of symmetry perpendicular to the plane of the annulus is :

- (a) $\frac{1}{2} M \frac{(b^4 + a^4)}{(b^2 - a^2)}$ (b) $\frac{1}{2} M \pi (b^2 - a^2)$ (c) $\frac{1}{2} M (b^2 - a^2)$ (d) $\frac{1}{2} M (b^2 + a^2)$

Soln. Consider a thin circular disc of radius 'r' and thickness 'dr' with 'ρ' as density.

$$I = \int r^2 dm = \int_a^b r^2 \rho 2\pi r dr = 2\pi \rho \int_a^b r^3 dr = \frac{2\pi \rho}{4} (b^4 - a^4)$$



Now, $M = \pi\rho(b^2 - a^2)$

Or $\rho = \frac{M}{\pi(b^2 - a^2)} \Rightarrow I = \frac{\pi}{2} \frac{M(b^4 - a^4)}{\pi(b^2 - a^2)} = \frac{M}{2}(b^2 + a^2)$

Correct answer is (d).

56. The principal value of the real integral $I = \int_{-3}^{+3} \frac{dx}{x^2 + 3x + 2}$ is:

- (a) $\frac{3\pi}{2}$ (b) $\ln\left(\frac{2}{5}\right)$ (c) ∞ (d) 0

Soln. $I = \int_{-3}^{+3} \frac{dx}{(x+1)(x+2)}$, the integrand blows up at $x = -1$, and at $x = 2$ but its principal value can be

calculated by using contour integration, $I = \oint \frac{dz}{(z+1)(z+2)}$, $f(z) = \frac{1}{(z+1)(z+2)}$ has poles are on the real axis and lie at $z = -1$ and $z = -2$

$$\text{Res. } f(z = -1) = \lim_{z \rightarrow -1} (z+1) \frac{1}{(z+1)(z+2)} = 1$$

$$\text{Res. } f(z = -2) = -1$$

$$I = 2\pi i \sum \text{Res.} = 0$$

Correct answer is (d).

57. The minimum energy of an electron (the rest mass of which is 0.5 MeV) that can emit Cerenkov radiation while passing through water (of refractive index 1.5) is approximately

- (a) 1.0 MeV (b) 3.0 MeV (c) 0.6 MeV (d) 0.5 MeV

Soln. $E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = \frac{0.5 \text{ MeV}}{\sqrt{1 - \frac{1}{n^2}}} = 0.67 \text{ MeV}$ (Since, $v = \frac{c}{n}$)

$E = 0.67 \text{ MeV}$ which is close to the approximate answer given.

Correct answer is (c).

58. A heater and a thermocouple are used to measure and control temperature T of a sample at $T_0 = 250^\circ \text{C}$. A feedback circuit supplies power 'P' to the heater according to the equation

$$P = P_0 + G(T_0 - T) - D \frac{dT}{dt}$$

with appropriately tuned values of the coefficients G and D . In order to maintain temperature stability in the presence of an external heat perturbation which causes small but rapid fluctuations of temperature, it is necessary to

- (a) decrease D (b) increase D (c) decrease G (d) increase G

Soln. $P = P_0 + G(T_0 - T) - D \frac{dT}{dt} \Rightarrow \frac{dP}{dt} = -G \frac{dT}{dt} - D \frac{d^2T}{dt^2} = 0$

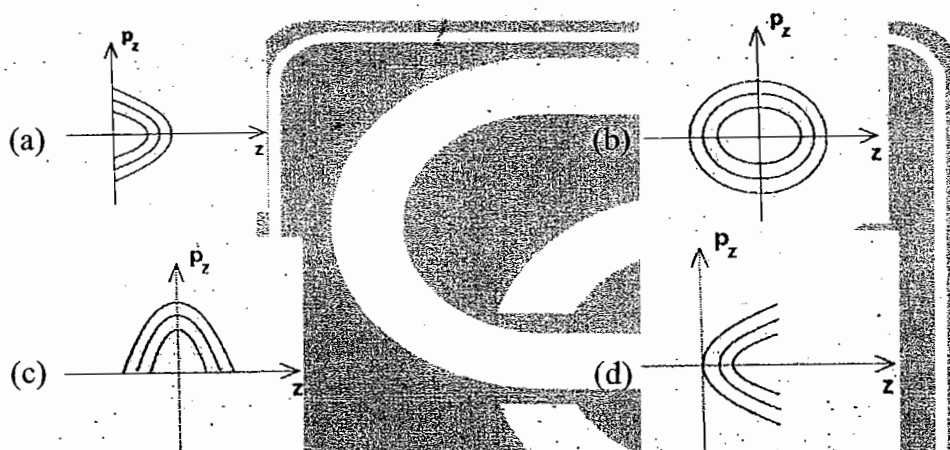
$$\Rightarrow G \frac{dT}{dt} + D \frac{d^2T}{dt^2} = 0 \quad \Rightarrow \frac{d}{dt} \left(G + D \frac{dT}{dt} \right) = 0$$

$$\Rightarrow G + D \frac{dT}{dt} = \text{constant.}$$

If $\frac{dT}{dt}$ increases, D must decrease.

Correct answer is (a)

59. The trajectory on the zp_z -plane (phase-space trajectory) of a ball bouncing perfectly elastically off a hard surface at $z = 0$ is given by approximately by (neglect friction) :



Soln. Going upward, $z = 0$, p_z = maximum, while going downward, $z = 0$, p_z = maximum in -ve p_z direction.

Correct answer is (a).

60. According to the shell model the spin and parity of the two nuclei $^{125}_{51}\text{Sb}$ and $^{89}_{38}\text{Sr}$ are, respectively.

(a) $\left(\frac{5}{2}\right)^+$ and $\left(\frac{5}{2}\right)^+$ (b) $\left(\frac{5}{2}\right)^+$ and $\left(\frac{7}{2}\right)^+$ (c) $\left(\frac{7}{2}\right)^+$ and $\left(\frac{5}{2}\right)^+$ (d) $\left(\frac{7}{2}\right)^+$ and $\left(\frac{7}{2}\right)^+$

Soln. $^{125}_{51}\text{Sb} \Rightarrow$ odd proton

Shell model energy levels, $1s_{1/2} 1p_{3/2} 1p_{1/2} 1d_{5/2} 2s_{1/2} 1d_{3/2} 1f_{7/2} 2p_{3/2} 1f_{5/2} 2p_{1/2} 1g_{9/2}$

Therefore, $^{125}_{51}\text{Sb}$ have odd proton, last proton will go to $1g_{7/2} (l=4)$

So, Spin parity $\Rightarrow \left(\frac{7}{2}\right)^+$

For $^{89}_{38}\text{Sr}$: 51 odd Neutron, last neutron will go to $2d_{5/2} (l=2)$

So, spin parity $\Rightarrow (5/2)^+$

So then are is $(7/2)^+$ and $(5/2)^+$

Correct answer is (c).

61. The wave function of a particle at time $t = 0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$, where $|u_1\rangle$ and $|u_2\rangle$ are the normalized eigenstates with eigenvalues E_1 and E_2 respectively, ($E_2 > E_1$). The shortest time after which $|\psi(t)\rangle$ will become orthogonal to $|\psi(0)\rangle$ is

- (a) $\frac{-\hbar\pi}{2(E_2 - E_1)}$ (b) $\frac{\hbar\pi}{E_2 - E_1}$ (c) $\frac{\sqrt{2}\hbar\pi}{E_2 - E_1}$ (d) $\frac{2\hbar\pi}{E_2 - E_1}$

Soln. $|\psi(0)\rangle = \frac{1}{\sqrt{2}}[|u_1\rangle + |u_2\rangle] \Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}[e^{-iE_1t/\hbar}|u_1\rangle + e^{-iE_2t/\hbar}|u_2\rangle]$ using $\langle\psi(0)|\psi(t)\rangle = 0$

$$\Rightarrow \frac{1}{2}\left(e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right) = 0 \Rightarrow e^{-iE_1t/\hbar} = -e^{-iE_2t/\hbar} \Rightarrow e^{-i(E_2 - E_1)t/\hbar} = -1 = e^{-i\pi} \text{ (for minimum time)}$$

$$\Rightarrow \frac{(E_2 - E_1)t}{\hbar} = \pi \Rightarrow t = \frac{\pi\hbar}{E_2 - E_1}$$

Correct answer is (b).

62. A gas of N non-interacting particles is in thermal equilibrium at temperature T . Each particle can be in any of the possible non-degenerate states of energy 0 , 2ϵ and 4ϵ . The average energy per particle of the gas, when $\beta\epsilon \ll 1$, is

- (a) 2ϵ (b) 3ϵ (c) $2\epsilon/3$ (d) ϵ

Soln.

$$\begin{aligned} & \frac{0 \cdot e^{-0/kT} + 2\epsilon \cdot e^{-2\epsilon/kT} + 4\epsilon \cdot e^{-4\epsilon/kT}}{e^{-0/kT} + e^{-2\epsilon/kT} + e^{-4\epsilon/kT}} = \frac{2\epsilon [e^{-2\epsilon/kT} + 2e^{-4\epsilon/kT}]}{1 + e^{-2\epsilon/kT} + e^{-4\epsilon/kT}} \end{aligned}$$

Since, $\beta\epsilon \ll 1$, $\frac{\epsilon}{kT} \ll 1 \Rightarrow$ we retain only the first term of exponentials.

$$\text{Therefore, } \langle E \rangle = \frac{2\epsilon(1+2)}{(1+1+1)} = 2\epsilon$$

Correct answer is (a)

63. The first few terms in the Taylor series expansion of the function $f(x) = \sin x$ around $x = \frac{\pi}{4}$ are

(a) $\frac{1}{\sqrt{2}} \left[1 + \left(x - \frac{\pi}{4}\right) + \frac{1}{2!} \left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!} \left(x - \frac{\pi}{4}\right)^3 \dots \right]$

(b) $\frac{1}{\sqrt{2}} \left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!} \left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!} \left(x - \frac{\pi}{4}\right)^3 \dots \right]$

(c) $\left[\left(x - \frac{\pi}{4}\right) - \frac{1}{3!} \left(x - \frac{\pi}{4}\right)^3 \dots \right]$

(d) $\frac{1}{\sqrt{2}} \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \dots \right]$

Soln. Taylor Series: $f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a)$

$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x$

$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\therefore f(x) = \frac{1}{\sqrt{2}} \left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!} \left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!} \left(x - \frac{\pi}{4}\right)^3 + \dots \right]$

Correct answer is (b)

64. A one-dimensional chain consists of a set of N rods each of length a . When stretched by a load, each rod can align either parallel or perpendicular to the length of the chain. The energy of a rod is $-\epsilon$ when aligned parallel to the length of the chain and is $+\epsilon$ when perpendicular to it. When the chain is in thermal equilibrium at temperature T , its average length is:

(a) $Na/2$

(b) Na

(c) $Na / (1 + e^{-2\epsilon/k_B T})$

(d) $Na / (1 + e^{-2\epsilon/k_B T})$

Soln. The mean length of the segment is $\ell = \frac{a e^{\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}} = \frac{a}{1 + e^{-2\beta\epsilon}}$

Average length $= \frac{Na}{(1 + e^{-2\epsilon/k_B T})}$

Correct answer is (c).

65. If the hyperfine interaction in an atom is given by $H = a \vec{S}_e \cdot \vec{S}_p$ Where $\vec{S}_e$ and $\vec{S}_p$ denote the electron and proton spins, respectively, the splitting between the $^3S_1$ and $^1S_0$ state is

(a) $a\hbar^2 / \sqrt{2}$

(b) $a\hbar^2$

(c) $a\hbar^2 / 2$

(d) $2a\hbar^2$

Soln. $H = a \frac{[S^2 - S_e^2 - S_p^2]}{2} = \frac{a}{2} \left[s(s+1)\hbar^2 - \frac{1}{2} \times \frac{3}{2} \hbar^2 - \frac{1}{2} \times \frac{3}{2} \hbar^2 \right]$

Splitting, $\Delta E = E_{s=1} - E_{s=0}$

(For singlet $s = 0$, For triplet $s = 1$)

$$= \frac{a}{2} \left(2\hbar^2 - \frac{3}{4}\hbar^2 \right) - \frac{a}{2} \left[0 - \frac{3}{4}\hbar^2 \right]$$

$$= \frac{a}{2} \times 2\hbar^2 - \frac{a}{2} \times \frac{3}{4}\hbar^2 + \frac{a}{2} \times \frac{3}{4}\hbar^2 = a\hbar^2$$

Correct answer is (b).

66. Consider a solenoid of radius R with n turns per unit length, in which a time dependent current $I = I_0 \sin \omega t$ (where $\omega R/c \ll 1$) flows. The magnitude of the electric field at a perpendicular distance $r < R$ from the axis of symmetry of the solenoid, is:

- (a) 0 (b) $\frac{1}{2r} \omega \mu_0 n I_0 R^2 \cos \omega t$
- (c) $\frac{1}{2} \omega \mu_0 n I_0 r \sin \omega t$ (d) $\frac{1}{2} \omega \mu_0 n I_0 r \cos \omega t$

Soln. $\int \vec{E} d\vec{l} = \varepsilon = -\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s}$ ϕ is the flux.

For solenoid, $B = \mu_0 n I = \mu_0 n I_0 \sin \omega t$

$$\Rightarrow E \cdot 2\pi r = -\frac{\partial}{\partial t} \mu_0 n I_0 \sin \omega t \pi r^2 = (-) \mu_0 n I_0 r^2 \frac{\partial}{\partial t} (\sin \omega t)$$

$$\text{Hence, } \Rightarrow |E| = \frac{1}{2} r \mu_0 n I_0 \omega \cos \omega t$$

Correct answer is (d).

67. The difference in the Coulomb energy between the mirror nuclei $^{49}_{24}\text{Cr}$ and $^{49}_{25}\text{Mn}$ is 6.0 MeV. Assuming that the nuclei have a spherically symmetric charge distribution and that e^2 is approximately 1.0 MeV-fm, the radius of the $^{49}_{25}\text{Mn}$ nucleus is

- (a) $4.9 \times 10^{-13} \text{ m}$ (b) $4.9 \times 10^{-15} \text{ m}$ (c) $5.1 \times 10^{-13} \text{ m}$ (d) $5.1 \times 10^{-15} \text{ m}$

Soln. Coulomb energy of uniformly charged sphere of radius, $R = \frac{3}{5} \frac{Z^2 e^2}{R}$

$$\therefore \frac{3}{5} \frac{(Z_1^2 - Z_2^2) e^2}{R} = 6.0 \text{ MeV}$$

$$\Rightarrow \frac{3}{5} \frac{(25^2 - 24^2)}{R} [1.0 \text{ MeV} \cdot \text{fm}] = 6.0 \text{ MeV}, [e^2 = 1.0 \text{ MeV} \cdot \text{fm}]$$

$$\Rightarrow R = 4.9 \text{ fm} = 4.9 \times 10^{-15} \text{ m}$$

Correct answer is (b).

68. The ratio of intensities of the D_1 and D_2 lines of sodium at high temperature

- (a) 1:1 (b) 2:3 (c) 1:3 (d) 1:2

Soln. D_1 line: $2P_{1/2} \rightarrow 2S_{1/2}$; D_2 line: $2P_{3/2} \rightarrow 2S_{1/2}$

The intensities will depend on the population of the levels

At high temperature ($kT > \Delta E$) the intensity ratio is determined by the ratio of degeneracies of

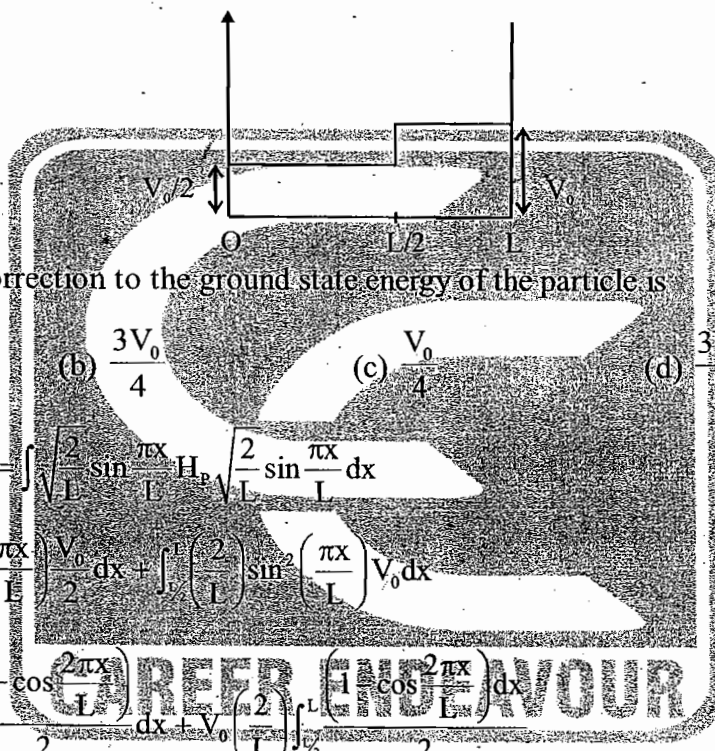
$$2P_{3/2} \text{ and } 2P_{1/2}$$

$$\frac{D_2}{D_1} = \frac{2J_2 + 1}{2J_1 + 1} = \frac{3 + 1}{1 + 1} = 2:1$$

$\therefore$ Ratio of $D_1:D_2$ is 1:2

Correct answer is (d).

69. A constant perturbation as shown in the figure below acts on a particle of mass m confined in a infinite potential well between 0 and L .



The first-order correction to the ground state energy of the particle is

(a) $\frac{V_0}{2}$

(b) $\frac{3V_0}{4}$

(c) $\frac{V_0}{4}$

(d) $\frac{3V_0}{2}$

Soln. $E'_0 = \langle \psi_0 | H_p | \psi_0 \rangle = \int_0^L \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} H_p \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} dx$

$$= \int_0^{L/2} \left(\frac{2}{L} \right) \sin^2 \left(\frac{\pi x}{L} \right) \frac{V_0}{2} dx + \int_{L/2}^L \left(\frac{2}{L} \right) \sin^2 \left(\frac{\pi x}{L} \right) V_0 dx$$

$$= \frac{V_0}{2} \times \frac{2}{L} \int_0^{L/2} \left(1 - \cos \frac{2\pi x}{L} \right) dx + V_0 \left(\frac{2}{L} \right) \int_{L/2}^L \left(1 - \cos \frac{2\pi x}{L} \right) dx$$

$$= \frac{V_0}{2L} \left[x - \frac{\sin \frac{2\pi x}{L}}{\frac{2\pi}{L}} \right]_0^{L/2} + \frac{V_0}{L} \left[x - \frac{\sin \frac{2\pi x}{L}}{\frac{2\pi}{L}} \right]_{L/2}^L = \frac{V_0}{2L} \times \frac{L}{2} + \frac{V_0}{L} \times \frac{L}{2} = \frac{V_0}{4} + \frac{V_0}{2} = \frac{3V_0}{4}$$

Correct answer is (b).

70. An atom of mass M can be excited to a state of mass $M + \Delta$ by photon capture. The frequency of a photon which can cause this transition is:

(a) $\Delta c^2 / 2h$

(b) $\Delta c^2 / h$

(c) $\Delta^2 c^2 / 2Mh$

(d) $\Delta(\Delta + 2M)c^2 / 2Mh$

Soln. $Mc^2 + h\nu = [(M + \Delta)^2 c^4 + c^2 p^2]^{1/2} = [(M^2 + \Delta^2 + 2M\Delta)c^4 + c^2 p^2]^{1/2}$

$$h\nu = [M^2 c^4 + \Delta^2 c^4 + 2M\Delta c^4 + c^2 p^2]^{1/2} - Mc^2$$

$$= Mc^2 \left[1 + \frac{\Delta^2}{M^2} + \frac{2\Delta}{M} + \frac{p^2}{M^2 c^2} \right]^{\frac{1}{2}} - Mc^2$$

Neglecting $P \ll Mc$ we get,

$$= Mc^2 \left[1 + \frac{\Delta^2}{M^2} + 2 \frac{\Delta}{M} \right]^{\frac{1}{2}} - Mc^2 = Mc^2 \left[1 + \frac{1}{2} \left(\frac{\Delta^2}{M^2} + 2 \frac{\Delta}{M} \right) \right] - Mc^2 = Mc^2 \left(\frac{\Delta}{M} + \frac{1}{2} \frac{\Delta^2}{M^2} \right)$$

$$\text{Therefore, } v = \frac{\Delta c^2 (2M + \Delta)}{2 M h}$$

Correct answer is (d).

71. The excitation of a three-dimensional solid are bosonic in nature with their frequency ω and wave-number k are related by $\omega \propto k^2$ in the large wavelength limit. If the chemical potential is zero, the behaviour of the specific heat of the system at low temperature is proportional to

(a) $T^{1/2}$

(b) T

(c) $T^{3/2}$

(d) T^3

Soln. $\omega \propto k^2$; $\hbar \omega \propto (\hbar k)^2$

$$\Rightarrow \epsilon \propto p^2$$

$$\text{Density of states } D(\epsilon) = \frac{4\pi V}{h^3} p^2 dp$$

$$\Rightarrow \epsilon \sim p^2 \Rightarrow d\epsilon \sim p dp \Rightarrow p \sim \frac{d\epsilon}{\sqrt{\epsilon}}$$

$$D(\epsilon) \sim \sqrt{\epsilon}; \quad E \propto \int_0^\infty \epsilon D(\epsilon) d\epsilon / (e^{\beta\epsilon} - 1)$$

$$\beta\epsilon = x; \quad \epsilon^{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}$$

$$\Rightarrow E \propto \int_0^\infty \frac{x^{\frac{1}{2}} dx}{e^x - 1} \frac{1}{\beta^{\frac{1}{2}}}$$

$$E \propto T^{\frac{1}{2}} (\because \text{integral} = \text{constant}) \Rightarrow C_v \propto T^{\frac{1}{2}}$$

Correct answer is (c).

72. Gas molecules of mass 'm' are confined in a cylinder of radius R and height L (with $R \gg L$) kept vertically in the Earth's gravitational field. The average energy of the gas at low temperatures (such that $mgL \gg k_B T$) is given by

(a) $Nk_B T/2$

(b) $3Nk_B T/2$

(c) $2Nk_B T$

(d) $5Nk_B T/2$

Soln. Average energy = (Number of degrees of freedom) $\times \frac{1}{2} kT$

For translational motion, number of degrees of freedom = 3.

$$\langle T \rangle = \frac{3}{2} kT, \quad V = mgz \text{ (Only one degree of freedom)}$$

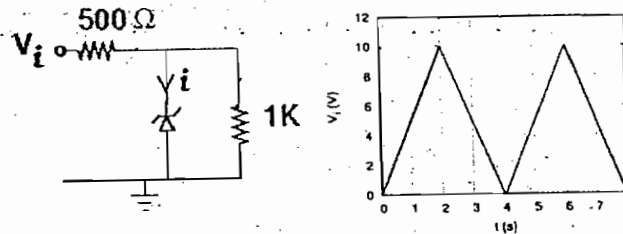
According to Virial theorem,

$$\langle V \rangle = kT \Rightarrow \langle E \rangle = \langle T \rangle + \langle V \rangle = \frac{5}{2} kT \text{ (for 1 particle)}$$

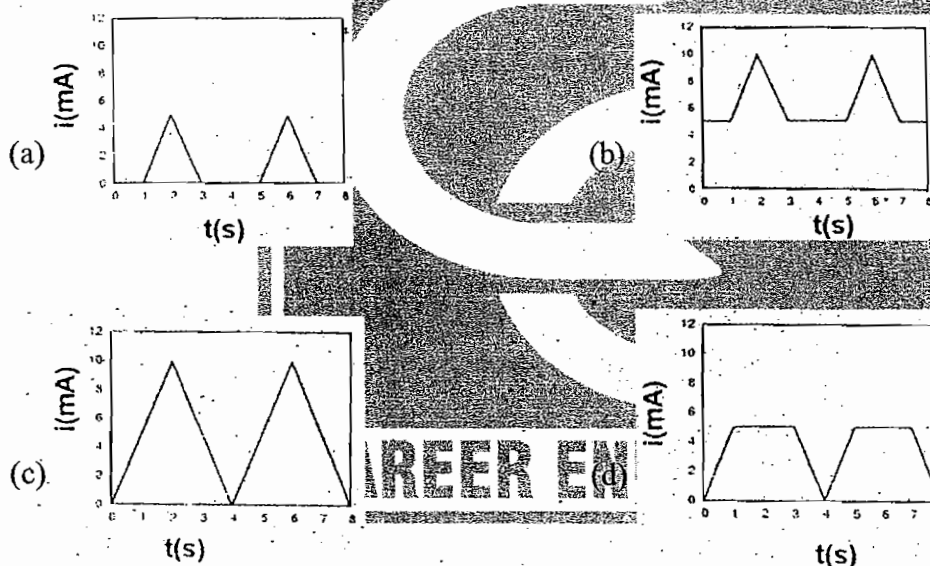
For N-particles, $\langle E \rangle = \frac{5}{2} NkT$

Correct answer is (d).

73. The figure below shows a voltage regulator utilizing a Zener diode of breakdown voltage 5V and a positive triangular wave input of amplitude 10V.



For $V_i > 5V$, the Zener regulates the output voltage by channeling the excess current through it self. Which of the following waveforms shows the current ' i ' passing through the Zener diode?



Soln. [As Zener diode is not ideal]

When $V_i < 5$ [upto $t = 1$ sec, $3 < t < 5$]

Then Zener is reverse biased hence, very-2 small current will pass through zener diode.

Then $i =$ very small fixed current will flow.

Now, when $V_i > 5$.

Then Zener comes in breakdown then maximum current will flows through it.

$1 K\Omega$ will be short circuited at 1 sec.

$$i = \frac{5}{500} = 10 \text{ mA}$$

which will rise steadily as it is not ideal.

Correct answer is (a).

74. A constant electric current I in an infinitely long straight wire is suddenly switched on at $t = 0$. The vector potential at a perpendicular distance r from the wire is given by

$$\vec{A} = \frac{\hat{k}\mu_0 I}{2\pi} \ln \left[\frac{1}{r} \left(ct + \sqrt{c^2 t^2 - r^2} \right) \right]. \text{ The electric field at a distance } r (< ct) \text{ is:}$$

- (a) 0 (b) $\frac{\mu_0 I}{2\pi t} \frac{1}{\sqrt{2}} (\hat{i} - \hat{j})$ (c) $\frac{c\mu_0 I}{2\pi \sqrt{c^2 t^2 - r^2}} \frac{1}{\sqrt{2}} (\hat{i} + \hat{j})$ (d) $-\frac{c\mu_0 I}{2\pi \sqrt{c^2 t^2 - r^2}} \hat{k}$

Soln. $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$, since $\vec{A}$ is in $\hat{k}$ -direction, $\vec{E}$ is also in $\hat{k}$ direction

Correct answer is (d).

75. Monochromatic light of wavelength 660 nm and intensity 100 mW/cm² falls on a solar cell of area 30 cm². The conversion efficiency of the solar cell is 10%. If each converted photon results in an electron-hole pair, what is the maximum circuit current supplied by the solar cell? (Take $h = 6.6 \times 10^{-34}$ J-s, $c = 3 \times 10^8$ m/s and $e = 1.6 \times 10^{-19}$ C)

- (a) 160 mA (b) 320 mA (c) 1600 mA (d) 3200 mA

Soln. Intensity fallen = 100 mW $\times$ 30 = 3 watt in the cell.
If the efficiency is 10% Then, 0.3 watt power is carried by the current.

$$\text{The maximum potential generated} = \frac{hc}{\lambda} = \frac{1.24 \times 10^{-6}}{660 \times 10^{-9}} = 1.88 \text{ eV}$$

$$\text{Therefore, the maximum current } I = \frac{P}{V} = \frac{3}{1.88} = 1.6 \text{ A} = 160 \text{ mA}$$

Correct answer is (a).

CAREER ENDEAVOUR



CSIR-UGC-NET/JRF- JUNE - 2012

PHYSICAL SCIENCES

Part-B

21. A vector perpendicular to any vector that lies on the plane defined by $x + y + z = 5$, is

- (a) $\hat{i} + \hat{j}$ (b) $\hat{j} + \hat{k}$ (c) $\hat{i} + \hat{j} + \hat{k}$ (d) $2\hat{i} + 3\hat{j} + 5\hat{k}$

Soln. Any vector perpendicular to any vector lies on $x + y + z = 5$ is $\hat{n} = \vec{\nabla} \phi = \hat{i} + \hat{j} + \hat{k}$

Correct answer is (c).

22. The eigenvalues of the matrix, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ are

- (a) (1, 4, 9) (b) (0, 7, 7) (c) (0, 1, 13) (d) (0, 0, 14)

Soln. Eigen values of a matrix have rows or columns that are scalar multiple of a particular row or column respectively are $\text{Tr}(A)$, 0, 0, ...

Eigen values of the given matrix $\lambda = 14, 0, 0$

Correct answer is (d).

23. The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $1 \leq |z| \leq 2$, and around $z = 1$ is

(a) $\frac{1}{2} \left[1 + z + z^2 + z^3 + \dots \right] \left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots \right]$

(b) $\frac{1}{1-z} + z - (1-z)^2 + (1-z)^3 + \dots$

(c) $\frac{1}{z^2} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right] \left[1 + \frac{2}{z} + \frac{4}{z^2} + \dots \right]$

(d) $2(z-1) + 5(z-1)^2 + 7(z-1)^3 + \dots$

Soln. $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{1-z} + \frac{1}{z-2} = \frac{1}{1-z} + \frac{1}{(z-1)-1}$ [Taking $z-1 = u$]

$$= -\frac{1}{u} + \frac{1}{u-1} = -\frac{1}{u} - (1-u)^{-1} = -\frac{1}{u} - 1 - u - u^2 \dots$$

$$= \frac{1}{1-z} - z - (z-1)^2 - (z-1)^3 + \dots = \frac{1}{1-z} - z - (1-z)^2 + (1-z)^3 + \dots$$

Only the principal part $\frac{1}{1-z}$ agree with option (b).

[Analytic part of the problem has wrong sign.

Correct answer is (b).

24. The radioactive decay of a certain material satisfies Poisson statistics with a mean rate of λ per second, what should be the minimum duration of counting (in seconds) so that the relative error is less than 1% ?
- (a) $100/\lambda$ (b) $10^4/\lambda^2$ (c) $10^4/\lambda$ (d) $1/\lambda$

Soln. Accuracy = 10^{-2} ; Mean rate = λs^{-1} ; Minimum duration of counting = $\frac{(\text{Accuracy})^2}{\lambda s^{-1}} = \frac{10^{-4}}{\lambda} s$

Correct answer is (c).

25. Let $u(x, y) = x + \frac{1}{2}(x^2 - y^2)$ be the real part of an analytic function $f(z)$ of the complex variable $z = x + iy$. the imaginary part of $f(z)$ is
- (a) $y + xy$ (b) xy (c) y (d) $y^2 - x^2$

Soln. $u(x, y) = x + \frac{1}{2}(x^2 - y^2)$; $f(z) = u(x, y) + iv(x, y)$

Cauchy Reimann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = 1 + x \Rightarrow \frac{\partial v}{\partial y} = 1 + x$

$$\Rightarrow v = y + xy + g(x) \quad \dots (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = -y$$

$$\Rightarrow \frac{\partial v}{\partial x} = y \Rightarrow v = xy + h(y) \quad \dots (2)$$

Compare (1) & (2), taking $h(y) = y$, and $g(x) = 0 \Rightarrow v = y + xy$

Correct answer is (a).

26. Let $y(x)$ be a continuous real function in the range 0 and 2π , satisfying the inhomogeneous differential equation

$$\sin x \frac{d^2 y}{dx^2} + \cos x \frac{dy}{dx} = \delta\left(x - \frac{\pi}{2}\right). \text{ The value of } dy/dx \text{ at the point } x = \pi/2$$

- (a) is continuous (b) has a discontinuity of 3
(c) has a discontinuity of 1/3 (d) has a discontinuity of 1

Soln. $\frac{d}{dx} \left[\sin x \frac{dy}{dx} \right] = \delta\left(x - \frac{\pi}{2}\right) \Rightarrow \sin x \frac{dy}{dx} = \int \delta\left(x - \frac{\pi}{2}\right) dx = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x} \int \delta\left(x - \frac{\pi}{2}\right) dx$$

$$\text{For } x = \frac{\pi}{2}, \frac{dy}{dx} = 1$$

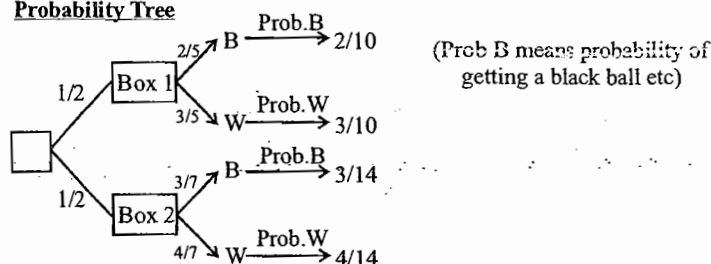
$$\text{For } x \neq \frac{\pi}{2}, \frac{dy}{dx} = 0$$

Therefore, $\frac{dy}{dx}$ has discontinuity of 1 at $x = \frac{\pi}{2}$

Correct answer is (d)

27. A ball is picked at random from one of two boxes that contain 2 black and 3 white and 3 black and 4 white balls respectively. What is the probability that it is white?
 (a) $34/70$ (b) $41/70$ (c) $36/70$ (d) $29/70$

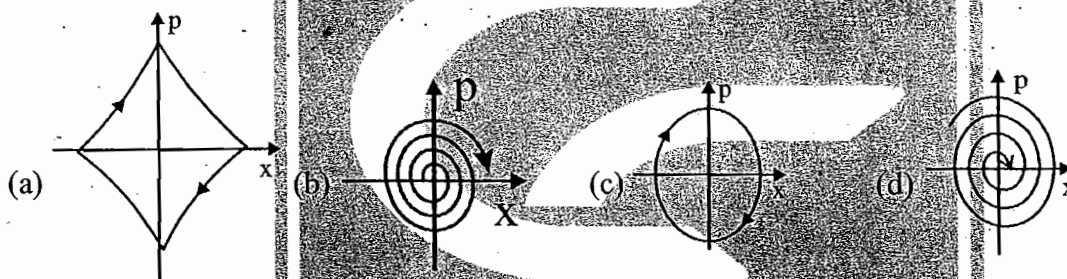
Soln. Probability Tree



$$\text{Probability of picking white ball} = \frac{3}{10} + \frac{4}{14} = \frac{41}{70}$$

Correct answer is (b).

28. The bob of a simple pendulum which undergoes small oscillations, is immersed in water. Which of the following figures best represents the phase space diagram for the pendulum?



Soln. It is a case of damped oscillator where the amplitude 'x' and momentum decreases towards zero. It is an inward spiral.

Correct answer is (d).

29. Two events, separated by a (spatial) distance $9 \times 10^9 \text{ m}$, are simultaneous in one inertial frame. The time interval between these two events in a frame moving with a constant speed $0.8c$ (where the speed of light $c = 3 \times 10^8 \text{ m/s}$) is:

- (a) 60 s (b) 40 s (c) 20 s (d) 0 s

Soln. $\Delta t' = \frac{\gamma v}{c^2} \Delta x = \frac{\gamma \beta}{c} \Delta x$. Since, $\left(\beta = 0.8, \gamma = \frac{1}{\sqrt{1-\beta^2}} \right) \Rightarrow \Delta t' = \frac{(0.8)}{(0.6)} \times \frac{9 \times 10^9 \text{ m}}{3 \times 10^8 \text{ m/s}} = 40 \text{ s}$

Correct answer is (b).

30. If the Lagrangian of a particle moving in one dimensions is given by $L = \frac{\dot{x}^2}{2x} - V(x)$, the Hamiltonian is:

- (a) $\frac{1}{2} x p^2 + V(x)$ (b) $\frac{\dot{x}^2}{2x} + V(x)$ (c) $\frac{1}{2} \dot{x}^2 + V(x)$ (d) $\frac{p^2}{2x} + V(x)$

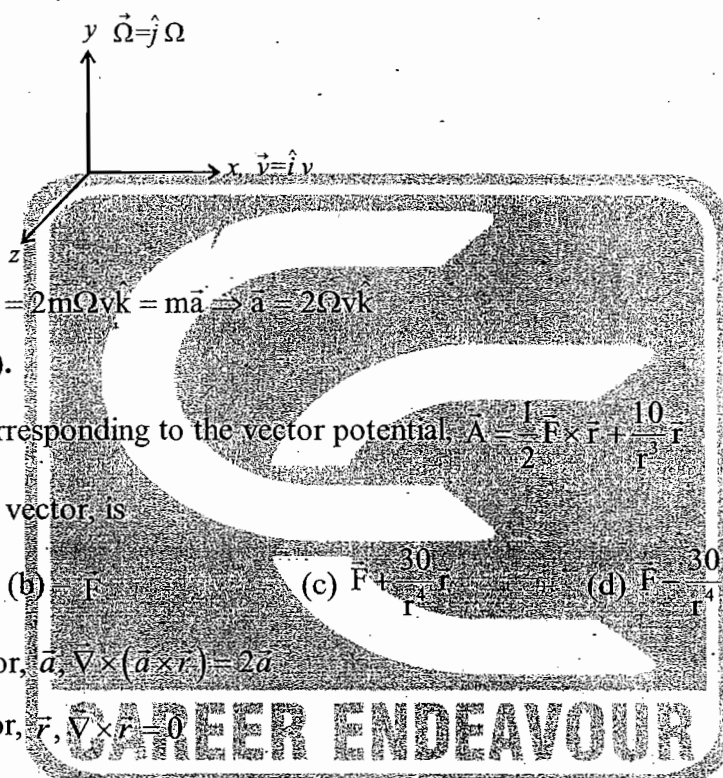
Soln. $L = \frac{\dot{x}^2}{2x} - v(x)$ and $H = p\dot{x} - L$; Now, $p = \frac{\partial L}{\partial \dot{x}} \Rightarrow p = \frac{2\dot{x}}{2x} = \frac{\dot{x}}{x} \Rightarrow \dot{x} = xp$.
 Therefore,

$$H = p(xp) - L = xp^2 - \left[\frac{(xp)^2}{2x} - V(x) \right] = xp^2 - \frac{x^2 p^2}{2x} + V(x) = xp^2 - \frac{1}{2} xp^2 + V(x) = \frac{xp^2}{2} + V(x)$$

Correct answer is (a)

31. A horizontal circular platform rotates with a constant angular velocity Ω directed vertically upwards. A person seated at the centre shoots a bullet of mass 'm' horizontally with speed 'v'. The acceleration of the bullet, in the reference frame of the shooter, is
 (a) $2v\Omega$ to his right (b) $2v\Omega$ to his left (c) $v\Omega$ to his right (d) $v\Omega$ to his left

Soln. Coriolis force = $2m(\vec{v} \times \vec{\Omega})$



$$\vec{F} = 2m(\hat{i} \times \hat{j})\Omega v \Rightarrow \vec{F} = 2m\Omega v \hat{k} = m\vec{a} \Rightarrow \vec{a} = 2\Omega v \hat{k}$$

Correct answer is (a).

32. The magnetic field corresponding to the vector potential, $\vec{A} = \frac{1}{2} \vec{F} \times \vec{r} + \frac{10}{r^3} \vec{r}$ where $\vec{F}$ is a constant vector, is
 (a) $\vec{F}$ (b) $-\vec{F}$ (c) $\vec{F} + \frac{30}{r^4} \vec{r}$ (d) $\vec{F} - \frac{30}{r^4} \vec{r}$

Soln. For any constant vector, $\vec{a}, \vec{\nabla} \times (\vec{a} \times \vec{r}) = 2\vec{a}$

For any position vector, $\vec{r}, \vec{\nabla} \times \vec{r} = 0$

$$\begin{aligned} \text{Therefore, } \vec{B} = \vec{\nabla} \times \vec{A} &= \frac{1}{2} \vec{\nabla} \times (\vec{F} \times \vec{r}) + 10 \vec{\nabla} \times \left(\frac{\vec{r}}{r^3} \right) \\ &= \vec{F} + 10 \left[\left(\vec{\nabla} \frac{1}{r^3} \right) \times \vec{r} + \frac{1}{r^3} (\vec{\nabla} \times \vec{r}) \right] = \vec{F} + 0 = \vec{F} \end{aligned}$$

Correct answer is (a).

33. An electromagnetic wave is incident on a water-air interface. The phase of the perpendicular component of the electric field, $E_{\perp}$, of the reflected wave into the water is found to remain the same for all angles of incidence. The phase of the magnetic field H.
 (a) does not change (b) changes by $3\pi/2$ (c) changes by $\pi/2$ (d) changes by π

Soln. Phase changes of magnetic field on reflection from a dielectric interface $\rho_{12} = -\left(\frac{n_2 - n_1}{n_2 + n_1} \right)$

$n_1 > n_2$ (water to air); $\rho_{12} = |\rho_{12}| e^{-i\pi}$. Phase changes by π .

Correct answer is (d).

34. The magnetic field at a distance R from a long straight wire carrying a steady current I is proportional to
 (a) IR (b) I/R^2 (c) I^2/R^2 (d) I/R

Soln. $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I \Rightarrow B \cdot 2\pi R = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi R}$

Correct answer is (d).

35. The component along an arbitrary direction $\hat{n}$, with direction cosines (n_x, n_y, n_z) , of the spin of a spin $\frac{1}{2}$ particle is measured. The result is:

- (a) 0 (b) $\pm \frac{\hbar}{2} n_z$ (c) $\pm \frac{\hbar}{2} (n_x, n_y, n_z)$ (d) $\pm \frac{\hbar}{2}$

Soln. $\vec{S} \cdot \hat{n} = S_x n_x + S_y n_y + S_z n_z$

$$= \begin{pmatrix} \cos \theta & \sin \theta \cos \phi - i \sin \theta \sin \phi \\ \sin \theta \cos \phi + i \sin \theta \sin \phi & -\cos \theta \end{pmatrix} \frac{\hbar}{2}$$

[Taking $\hat{n} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$]

Solving the eigenvalue equation, $|\vec{S} \cdot \hat{n} - \lambda I| = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$

Correct answer is (d).

36. A particle of mass m is in a cubic box of size a . The potential inside the box, $(0 \leq x < a, 0 \leq y < a, 0 \leq z < a)$ is zero and infinite outside. If the particle is in an eigenstate of energy $E = \frac{14\pi^2 \hbar^2}{2ma^2}$, its wave function is

- (a) $\psi = \left(\frac{2}{a}\right)^{3/2} \sin \frac{3\pi x}{a} \sin \frac{5\pi y}{a} \sin \frac{6\pi z}{a}$ (b) $\psi = \left(\frac{2}{a}\right)^{3/2} \sin \frac{7\pi x}{a} \sin \frac{4\pi y}{a} \sin \frac{3\pi z}{a}$
 (c) $\psi = \left(\frac{2}{a}\right)^{3/2} \sin \frac{4\pi x}{a} \sin \frac{8\pi y}{a} \sin \frac{2\pi z}{a}$ (d) $\psi = \left(\frac{2}{a}\right)^{3/2} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a} \sin \frac{3\pi z}{a}$

Soln. The energy of the particle $E = \frac{14\hbar^2 \pi^2}{2ma^2}$

The energy of the particle in cubic box of size a is $= \frac{\hbar^2 \pi^2}{2ma^2} (n_x^2 + n_y^2 + n_z^2)$

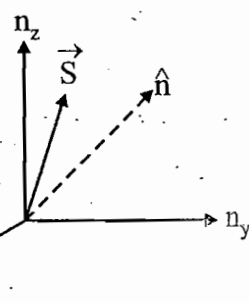
[where, n_x, n_y , and $n_z = 1, 2, 3, 4, \dots$]

For energy $E = \frac{14\hbar^2 \pi^2}{2ma^2}$, the possible values of $(n_x, n_y, n_z) = (1, 2, 3)$.

The wave function the particle $\psi_n = \left(\sqrt{\frac{2}{a}}\right)^3 \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right)$

On substituting the values of (n_x, n_y, n_z) $\psi = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right) \sin\left(\frac{3\pi z}{a}\right)$

Correct answer is (d).



37. Let $\Psi_{n\ell m}$ denote the eigenfunctions of a Hamiltonian for a spherically symmetric potential $V(r)$. The wavefunction $\Psi = \frac{1}{4}[\Psi_{210} + \sqrt{5}\Psi_{21-1} + \sqrt{10}\Psi_{211}]$ is an eigenfunction only of

(a) H, L^2 and L_z (b) H and L_z (c) H and L^2 (d) L^2 and L_z

Soln. The wavefunction $\Psi = \frac{1}{4}[\Psi_{210} + \sqrt{5}\Psi_{21-1} + \sqrt{10}\Psi_{211}]$

For spherically symmetric potential $H\Psi_{n\ell m} = E_n\Psi_{n\ell m}$, where E_n depends only on n and independent of ℓ and m .

$$L^2\Psi_{n\ell m} = \ell(\ell+1)\hbar^2\Psi_{n\ell m}; L_z\Psi_{n\ell m} = m\hbar\Psi_{n\ell m}$$

$$H\Psi = \frac{1}{4}[H\Psi_{210} + \sqrt{5}H\Psi_{21-1} + \sqrt{10}H\Psi_{211}] = \frac{E_n}{4}[\Psi_{210} + \sqrt{5}\Psi_{21-1} + \sqrt{10}\Psi_{211}] \text{ with } n=2$$

So Ψ is an eigen function of H .

$$L^2\Psi = \frac{1}{4}[L^2\Psi_{210} + \sqrt{5}L^2\Psi_{21-1} + \sqrt{10}L^2\Psi_{211}] = \frac{1}{4}[2\hbar^2\Psi_{210} + \sqrt{5}2\hbar^2\Psi_{21-1} + \sqrt{10}2\hbar^2\Psi_{211}] \text{ with } \ell=1.$$

$$= \frac{2\hbar^2}{4}[\Psi_{210} + \sqrt{5}\Psi_{21-1} + \sqrt{10}\Psi_{211}] = 2\hbar^2\Psi$$

$$L_z\Psi = \frac{1}{4}[L_z\Psi_{210} + \sqrt{5}L_z\Psi_{21-1} + \sqrt{10}L_z\Psi_{211}]$$

$$= \frac{1}{4}[0 \times \Psi_{210} + \sqrt{5}(-\hbar)\Psi_{21-1} + \sqrt{10}(\hbar)\Psi_{211}] = \frac{\hbar}{5}[\sqrt{5}\Psi_{21-1} + \sqrt{10}\Psi_{211}] \neq (\text{constant})\Psi$$

When we operate H, L^2 and L_z on a given wavefunction, then in the case of H and L^2 , there is no change in the form of wavefunction but in case of L_z , wavefunction changes, so the given wavefunction is the eigen function of H and L^2 only.

Soln. Correct answer is (c).

38. The commutator $[x^2, p^2]$ is

(a) $2i\hbar xp$ (b) $2i\hbar(xp + px)$ (c) $2i\hbar px$ (d) $2i\hbar(xp - px)$

Soln. $[x^2, p^2] = x[x, p^2] + [x, p^2]x$ [By using distributive property]

$$= xp[x, p] + x[x, p]p + p[x, p]x + [x, p]px = xp(i\hbar) + x(i\hbar)p + p(i\hbar)x + (i\hbar)px$$

$$= 2i\hbar xp + 2i\hbar px = 2i\hbar(xp + px)$$

Correct answer is (b).

39. Consider a system of non-interacting particles in d dimensions obeying the dispersion relation $\varepsilon = Ak^s$, where ε is the energy, k is the wavevector, 's' is an integer and A a constant. The density of states, $N(\varepsilon)$, is proportional to

(a) $\varepsilon^{\frac{1}{d}-1}$ (b) $\varepsilon^{\frac{d}{s}+1}$ (c) $\varepsilon^{\frac{d}{s}-1}$ (d) $\varepsilon^{\frac{s}{d}-s}$

Soln. The dispersion relation $\varepsilon = Ak^s$

The volume of sphere in d - dimension in k -space

$$= \frac{\pi^{d/2}}{d/2!} k^d \quad [\text{See appendix C on page 504 in Statistical Mechanics-II edition by R.K. Pathria}]$$

The number of microstates $(\Omega) \propto k^d$; $\Omega \propto \varepsilon^{d/s}$

The number of microstates in the range ε to $\varepsilon + d\varepsilon$ is proportional to $\left(\varepsilon^{\frac{d}{s}-1} \right) d\varepsilon$

The density of states $\propto \varepsilon^{\frac{d}{s}-1}$

Soln. Correct answer is (c).

40. The number of ways in which N identical bosons can be distributed in two energy levels, is
 (a) $N+1$ (b) $N(N-1)/2$ (c) $N(N+1)/2$ (d) N

Soln. The number of ways in which N identical bosons can be distributed in two energy levels

$$\text{Number of ways} = \frac{(N+2-1)!}{N!(2-1)!} = \frac{(N+1)!}{N!} = \frac{(N+1)N!}{N!} = N+1$$

Correct answer is (a).

41. The free energy of a gas N particles in a volume V and at a temperature T is $F = NK_B T \ln [a_0 V (k_B T)^{5/2} / N]$, where a_0 is a constant and k_B denotes the Boltzmann constant. The internal energy of gas is

(a) $\frac{3}{2} Nk_B T$

(b) $\frac{5}{2} Nk_B T$

(c) $Nk_B T \ln [a_0 V (k_B T)^{5/2} / N] - \frac{3}{2} Nk_B T$

(d) $Nk_B T \ln [a_0 V / (k_B T)^{5/2}]$

Soln. The Free energy, $F = NK_B T \ln [a_0 V (k_B T)^{5/2} / N]$; $F = -k_B T \ln Q$

$$\ln Q = -\ln [a_0 V (k_B T)^{5/2} / N]^N; \quad U = -\frac{\partial}{\partial \beta} \ln Q = k_B T^2 \frac{\partial}{\partial T} \ln Q; \quad \left| \begin{array}{l} \beta = \frac{1}{k_B T} \\ \frac{\partial}{\partial \beta} = \frac{-1}{k_B T^2} \frac{\partial}{\partial T} \end{array} \right.$$

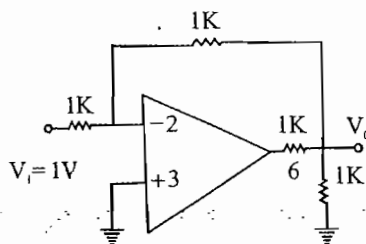
$$= k_B T^2 \frac{\partial}{\partial T} \left[-\ln \left\{ a_0 V (k_B T)^{5/2} / N \right\}^N \right] = k_B T^2 \frac{\partial}{\partial T} \left[-N \ln T^{5/2} - N \ln \left(\frac{a_0 V k_B}{N} \right) \right]$$

$$= -\frac{5N}{2} k_B T^2 \times \frac{1}{T} = -\frac{5}{2} Nk_B T$$

[Extra -ve sign is due to the wrong sign in the given question]

Correct answer is (b).

42. In the op-amp circuit shown in the figure below, the input voltage V_i is 1V. The value of the output V_o is



- (a) -0.33 V (b) -0.50 V (c) -1.00 V (d) -0.25 V

Soln. Apply Nodal Analysis:

$$\Rightarrow \frac{0-1}{1} + \frac{0-V_o}{1} = 0 \Rightarrow -1 + (-V_o) = 0 \Rightarrow V_o = -1V$$

Correct answer is (c).

43. An LED operates at 1.5 V and 5 mA in forward bias. Assuming an 80 % external efficiency of the LED, how many photons are emitted per second?

- (a) 5.0×10^{16} (b) 1.5×10^{16} (c) 0.8×10^{16} (d) 2.5×10^{16}

Soln. Power = $(VI) \times \text{efficiency} = (1.5 \text{ V})(5 \text{ mA})(0.8) \text{ watt} = nh\nu$.

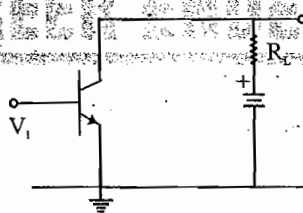
[E_g of semiconductor is not given but answer will be correct. If we use GaAs semiconductor.]

Semiconductor LED, $E_g = 1.43 \text{ eV} \Rightarrow nh\nu = 6 \text{ watt}$

$$\Rightarrow n = \frac{6}{1.43 \times 1.6 \times 10^{-19}} \therefore n = 2.5 \times 10^{16} / \text{sec}$$

Correct answer is (d).

44. The transistor in the given circuit has $h_{fe} = 35\Omega$ and $h_{ie} = 1000\Omega$. If the load resistance $R_L = 1000\Omega$, the voltage and current gain are, respectively.



- (a) $-35, +35$ (b) $35, -35$ (c) $35, -0.97$ (d) $0.98, -35$

Soln. $h_{fe} = 35\Omega$; $h_{ie} = 1000\Omega = R_i$; $R_L = 1000\Omega$

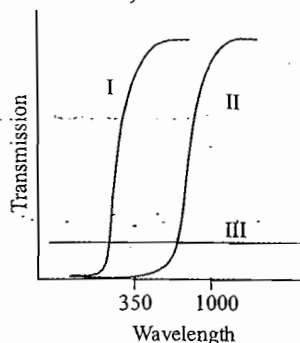
For common emitter voltage gain = $h_{fe} = -35 \Rightarrow R_i = h_{ie} = 1000\Omega$

$$A_i = \frac{i_L}{i_i} = \left(\frac{-i_o}{i_i} \right) = + \frac{h_{fe} i_i}{i_i} = +h_{fe} = +35$$

$$A_v = -A_i \left(\frac{R_c}{R_i} \right) = -A_i \left(\frac{1000}{1000} \right) = -35$$

Correct answer is (a).

45. The experimentally measured transmission spectra of metal, insulator and semiconductor thin films are shown in the figure. It can be inferred that I, II and III correspond respectively, to



- (a) Insulator, semiconductor and metal. (b) Semiconductor, metal and insulator
(c) Metal, semiconductor and insulator (d) Insulator, metal and semiconductor

Soln. Transmittance of metal does not depend on wavelength (λ). Hence III is metal.

Correct answer is (a).

46. The eigenvalues of the antisymmetric matrix, $A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$ where n_1, n_2 and n_3

are the

components of a unit vector, are

- (a) 0, i, -i

- (b) 0, 1, -1

- (c) 0, 1 + i, -1 - i

- (d) 0, 0, 0

Soln. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -n_3 & n_2 \\ n_3 & -\lambda & -n_1 \\ -n_2 & n_1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 + \lambda(n_1^2 + n_2^2 + n_3^2) = 0$$

$$\Rightarrow \lambda^3 + \lambda = 0$$

$$\Rightarrow \lambda = 0, i, -i$$

Correct answer is (a).

47. Which of the following limits exists?

(a) $\lim_{N \rightarrow \infty} \left(\sum_{m=1}^N \frac{1}{m} + \ln N \right)$

(b) $\lim_{N \rightarrow \infty} \left(\sum_{m=1}^N \frac{1}{m} - \ln N \right)$

(c) $\lim_{N \rightarrow \infty} \left(\sum_{m=1}^N \frac{1}{\sqrt{m}} - \ln N \right)$

(d) $\lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{1}{m}$

Soln. $S(N) = \sum_{m=1}^N \frac{1}{m} = \ln N + \gamma$

$N \rightarrow \infty$

where, $\gamma = 0.57721$ (Euler constant)

$$\therefore \lim_{N \rightarrow \infty} \left[\sum_{m=1}^N \frac{1}{m} - \ln N \right] = \ln N + \gamma - \ln N = \gamma$$

Correct answer is (b).

48. A bag contains many balls, each with a number painted on it. There are exactly n balls which have the number n (namely one ball with 1, two balls with 2, and so on until N balls with N on them). An experiment consists of choosing a ball at random, noting the number on it and returning it to the bag. If the experiment is repeated a large number of times, the average value of the number will tend to

- (a) $\frac{2N+1}{3}$ (b) $\frac{N}{2}$ (c) $\frac{N+1}{2}$ (d) $\frac{N(N+1)}{2}$

Soln. Take $N = 4$.

The configuration is

1

2	2
---	---

3	3	3
---	---	---

4	4	4	4
---	---	---	---

Sample space: $1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \sum_{n=1}^4 n^2$

Number of entries: $1 + 2 + 3 + 4 = 10 = \sum_{n=1}^4 n$, Average = $\frac{30}{10} = 3$

In general,

$$\text{Average value of the number} = \frac{\sum_{i=1}^N n_i^2}{\sum_{j=1}^N n_j} = \frac{\left[\frac{N(N+1)(2N+1)}{6} \right]}{\left[\frac{N(N+1)}{2} \right]} = \frac{2N+1}{3}$$

Correct answer is (a).

49. The value of the integral $\int_{-\infty}^{\infty} \frac{1}{t^2 - R^2} \cos\left(\frac{\pi t}{2R}\right) dt$ is
- (a) $-2\pi/R$ (b) $-\pi/R$ (c) π/R (d) $2\pi/R$

Soln. $\int_{-\infty}^{\infty} \frac{1}{t^2 - R^2} \cos\left(\frac{\pi t}{2R}\right) dt = \text{Real part} \int_{-\infty}^{\infty} \frac{1}{t^2 - R^2} \exp\left(\frac{\pi t}{2R}\right) dt$

$f(z) = \frac{1}{z^2 - R^2} \exp\left(\frac{\pi t}{2R}\right)$ has two poles at $z = +R, -R$

Residue at $(z = R) = \frac{1}{2R} e^{\frac{i\pi}{2}}$; Residue at $(z = -R) = -\frac{1}{2R} e^{-\frac{i\pi}{2}}$

Since the two poles are on real axis, therefore, required integral = Real part $(\pi i \times \text{sum of the residues})$

$$\Rightarrow I = \text{Real part} \left[\pi i \left(\frac{1}{2R} \times e^{\frac{i\pi}{2}} - \frac{1}{2R} \times e^{-\frac{i\pi}{2}} \right) \right] = -\frac{\pi}{R}$$

Correct option is (b).

50. The Poisson bracket $\{[r], [p]\}$ has the value

- (a) $|r||p|$ (b) $\hat{r} \cdot \hat{p}$ (c) 3 (d) 1

Soln. $\left\{ (x^2 + y^2 + z^2)^{1/2}, (p_x^2 + p_y^2 + p_z^2)^{1/2} \right\}$

$$\begin{aligned}
 &= \frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial x} \frac{\partial(p_x^2 + p_y^2 + p_z^2)^{1/2}}{\partial p_x} + \frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial y} \frac{\partial(p_x^2 + p_y^2 + p_z^2)^{1/2}}{\partial p_y} \\
 &\quad + \frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial z} \frac{\partial(p_x^2 + p_y^2 + p_z^2)^{1/2}}{\partial p_z} \\
 &= \left[\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x \right] \left[\frac{1}{2}(p_x^2 + p_y^2 + p_z^2)^{-1/2} \cdot 2p_x \right] + \\
 &\quad \left[\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y \right] \left[\frac{1}{2}(p_x^2 + p_y^2 + p_z^2)^{-1/2} \cdot 2p_y \right] + \left[\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2z \right] \left[\frac{1}{2}(p_x^2 + p_y^2 + p_z^2)^{-1/2} \cdot 2p_z \right] \\
 &= \frac{x}{r} \frac{p_x}{p} + \frac{y}{r} \frac{p_y}{p} + \frac{z}{r} \frac{p_z}{p} = \frac{xp_x + yp_y + zp_z}{rp} = \frac{\vec{r} \cdot \vec{p}}{rp} = \hat{r} \cdot \hat{p}
 \end{aligned}$$

Correct answer is (b).

51. Consider the motion of a classical particle in a one dimensional double-well potential

$V(x) = \frac{1}{4}(x^2 - 2)^2$. If the particle is displaced infinitesimally from the minimum the positive x-axis (and friction is neglected), then

- (a) the particle will execute simple harmonic motion in the right well with an angular frequency $\omega = \sqrt{2}$
 (b) the particle will execute simple harmonic motion in the right well with an angular frequency $\omega = 2$
 (c) the particle will switch between the right and left wells
 (d) the particle will approach the bottom of the right well and settle there

Soln. $V(x) = \frac{1}{4}(x^2 - 2)^2 \Rightarrow \frac{dV(x)}{dx} = \frac{1}{2}(x^2 - 2) \cdot (2x) = 0 \Rightarrow x = 0 \text{ and } x = \pm\sqrt{2}$

$$\left. \frac{d^2V(x)}{dx^2} \right|_{x=0} = 3x^2 - 2 \Big|_{x=0} = -2 \quad (\text{Discard this as this refers to maximum})$$

$$\left. \frac{d^2V(x)}{dx^2} \right|_{x=\pm\sqrt{2}} = 3 \times 2 - 2 = 4 \Rightarrow \omega^2 = 4 \text{ (for unit mass)} \Rightarrow \omega = 2$$

Correct answer is (b).

52. What is the proper time interval between the occurrence of two events if in one inertial frame the events are separated by 7.5×10^8 m and occur 6.5 s apart?

- (a) 6.50 s (b) 6.00 s (c) 5.75 s (d) 5.00 s

Soln. Proper time interval between two events

$$\Delta\tau = \sqrt{(\Delta t)^2 - \frac{(\Delta x)^2}{c^2} - \frac{(\Delta y)^2}{c^2} - \frac{(\Delta z)^2}{c^2}} = \sqrt{(\Delta t)^2 - \left(\frac{\Delta r}{c}\right)^2}$$

Given, $\Delta t = 6.5$ s; $\Delta r = 7.5 \times 10^8$ m; $\frac{\Delta r}{c} = \frac{7.5 \times 10^8 \text{ m}}{3.0 \times 10^8 \text{ m/s}} = 2.5$ s, we get $\Delta\tau = \sqrt{(6.5)^2 - (2.5)^2} = 6.0$ s

Correct answer is (b).

53. A free particle described by a plane wave and moving in the positive z-direction undergoes scattering by a potential $V(r) = \begin{cases} V_0 & \text{if } r \leq R \\ 0 & \text{if } r > R \end{cases}$. If V_0 changes to $2V_0$, keeping R fixed, then the differential scattering cross-section, in the Born approximation,
- (a) increases to four times the original value (b) increases to twice the original value
(c) decreases to half the original value (d) decreases to one fourth the original value

Soln. The scattering amplitude in the born approximation, $f(\theta, \phi) = -\frac{2m}{\hbar^2 K} \int_0^\infty r \sin Kr V(r) dr$

$$\text{With } K = 2k \sin(\theta/2) \Rightarrow f(\theta, \phi) = \frac{-2m}{\hbar^2 K} \left[\int_0^R r \sin Kr V_0 dr \right] = \frac{-2m}{\hbar^2 K} V_0 \int_0^R r \sin Kr dr$$

The differential cross - section

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2 = \left(\frac{2mV_0}{\hbar^2 K} \right)^2 \left[\int_0^R r \sin Kr dr \right]^2$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega} \right)_{2V_0} = 4 \left(\frac{d\sigma}{d\Omega} \right)_{V_0}$$

Correct answer is (a).

54. A variational calculation is done with the normalized trial wavefunction $\psi(x) = \frac{\sqrt{15}}{4a^{5/2}}(a^2 - x^2)$ for the one dimensional potential well

$$V(x) = \begin{cases} 0 & \text{if } |x| \leq a \\ \infty & \text{if } |x| > a \end{cases}$$

The ground state energy is estimated to be

- (a) $\frac{5h^2}{3ma^2}$ (b) $\frac{3h^2}{2ma^2}$ (c) $\frac{3h^2}{5ma^2}$ (d) $\frac{5h^2}{4ma^2}$

Soln. The trial wave function $\psi(x) = \frac{\sqrt{15}}{4a^{5/2}}(a^2 - x^2)$ and $V(x) = \begin{cases} 0 & \text{if } |x| \leq a \\ \infty & \text{if } |x| \geq a \end{cases}$

$$\langle H \rangle = \int \psi^* H \psi dx = \int_{-\infty}^{+\infty} \psi^* \left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) \psi dx$$

$$= \int_{-\infty}^{-a} 0 \left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) 0 dx + \int_{-a}^{+a} \frac{\sqrt{15}}{4a^{5/2}}(a^2 - x^2) \left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) \frac{\sqrt{15}}{4a^{5/2}}(a^2 - x^2) dx + \int_{+a}^{+\infty} 0 \left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) 0 dx$$

$$= \frac{15}{4 \times 4a^5} \left(\frac{-\hbar^2}{2m} \right) \int_{-a}^{+a} (a^2 - x^2)(-2) dx = \frac{15}{16a^5} \left(\frac{2\hbar^2}{2m} \right) \times 2 \int_0^a (a^2 - x^2) dx$$

$$= \frac{15}{8a^5} \left(\frac{\hbar^2}{m} \right) \left(a^2 x - \frac{x^3}{3} \right)_0^a = \frac{15}{8a^5} \left(\frac{\hbar^2}{m} \right) \times \frac{2}{3} a^3 = \frac{5}{4} \frac{\hbar^2}{ma^2}$$

Correct answer is (d).

55. A particle in one-dimension is in the potential

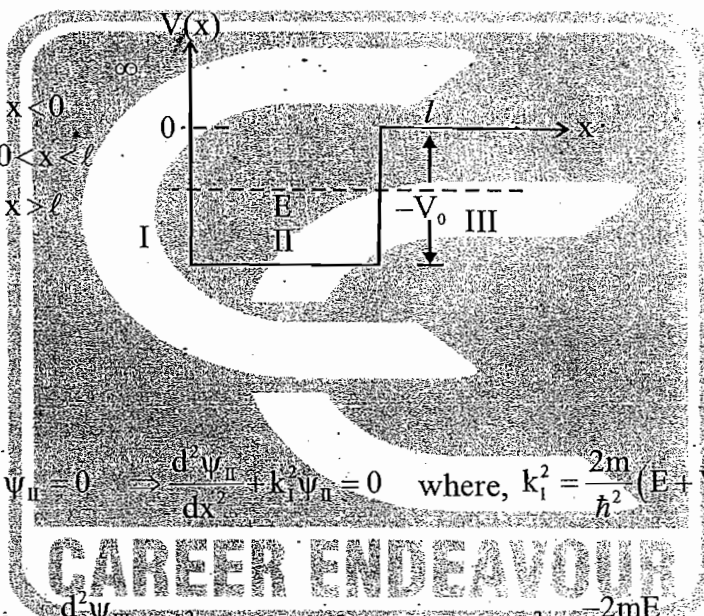
$$V(x) = \begin{cases} \infty & \text{if } x < 0 \\ -V_0 & \text{if } 0 \leq x \leq \ell \\ 0 & \text{if } x > \ell \end{cases}$$

If there is at least one bound state, the minimum depth of the potential is

(a) $\frac{\hbar^2 \pi^2}{8m\ell^2}$ (b) $\frac{\hbar^2 \pi^2}{2m\ell^2}$ (c) $\frac{2\hbar^2 \pi^2}{m\ell^2}$ (d) $\frac{\hbar^2 \pi^2}{m\ell^2}$

Soln.

$$V(x) = \begin{cases} \infty & \text{if } x < 0 \\ -V_0 & \text{if } 0 < x < \ell \\ 0 & \text{if } x > \ell \end{cases}$$



Region-I:

$$\psi_I = 0$$

Region-II:

$$\frac{d^2 \psi_{II}}{dx^2} + \frac{2m}{\hbar^2} (E + V_0) \psi_{II} = 0 \Rightarrow \frac{d^2 \psi_{II}}{dx^2} + k_1^2 \psi_{II} = 0 \quad \text{where, } k_1^2 = \frac{2m}{\hbar^2} (E + V_0)$$

Region-III:

$$\frac{d^2 \psi_{III}}{dx^2} + \frac{2mE}{\hbar^2} \psi_{III} = 0 \Rightarrow \frac{d^2 \psi_{III}}{dx^2} - k_2^2 \psi_{III} = 0, \quad \text{where, } k_2^2 = \frac{2mE}{\hbar^2}$$

Solve the above equation,

$$\psi_{II} = A \sin k_1 x + B \cos k_1 x$$

$$\psi_{III} = C e^{k_2 x} + D e^{-k_2 x}$$

For wavefunction to be finite at $x = \infty$, $C = 0$

For the wavefunction to be zero at $x = 0$, $B = 0$.

Therefore, $\psi_{II}(x) = D e^{-k_2 x}$

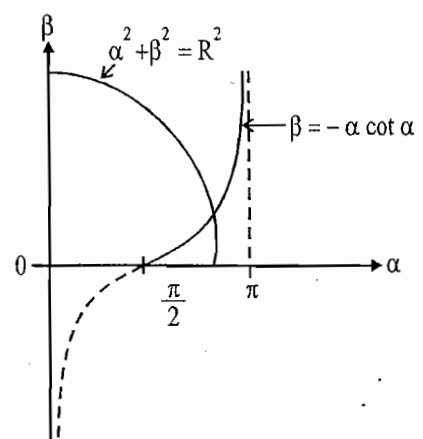
$$\psi_{III}(x) = A \sin k_1 x$$

Boundary condition, $\psi_I|_{x=0} = \psi_{II}|_{x=0} = 0 \Rightarrow B = 0$

$$\psi_{II}|_{x=\ell} = \psi_{III}|_{x=\ell} \Rightarrow A \sin k_1 \ell = D e^{-k_2 \ell} \quad (1)$$

$$\frac{d\psi_{II}}{dx} \Big|_{x=\ell} = \frac{d\psi_{III}}{dx} \Big|_{x=\ell} \quad (2)$$

$$k_1 A \cos k_1 \ell = -k_2 D e^{-k_2 \ell}$$



From equation (1) and (2)

$$-k_2 = k_1 \cot k_1 \ell \Rightarrow k_2 \ell = -k_1 \ell \cot k_1 \ell$$

(Taking $k_1 \ell = \alpha$, $k_2 \ell = \beta - \alpha \cot \alpha = \beta$)

$$\Rightarrow \alpha^2 + \beta^2 = \frac{2mV_0 \ell^2}{\hbar^2} = R^2 \text{ (equation of circle)}$$

To exist one bound state, $\frac{\pi}{2} \leq R < \pi$. The minimum value of $R = \frac{\pi}{2}$

$$\Rightarrow R^2 = \frac{\pi^2}{4} \Rightarrow \frac{2mV_0 \ell^2}{\hbar^2} = \frac{\pi^2}{4} \Rightarrow V_0 = \frac{\hbar^2 \pi^2}{8m\ell^2}$$

Correct answer is (a).

56. Which of the following is a self-adjoint operator in the spherical polar coordinate system (r, θ, ϕ) ?

(a) $-\frac{i\hbar}{\sin^2 \theta} \frac{\partial}{\partial \theta}$ (b) $-\frac{i\hbar}{\sin \theta} \frac{\partial}{\partial \theta}$ (c) $-\frac{i\hbar}{\sin \theta} \frac{\partial}{\partial \theta}$ (d) $-\frac{i\hbar \sin \theta}{\partial \theta}$

Soln. To calculate self adjoint we use condition

$$(\phi, \hat{A}\psi) = (\hat{A}\phi, \psi) \Rightarrow \int \phi^* \hat{A}\psi d\tau = \int (\hat{A}\phi)^* \psi d\tau$$

$$\int \phi^*(r, \theta, \phi) \left(-\frac{i\hbar}{\sin \theta} \frac{\partial}{\partial \theta} \right) \psi(r, \theta, \phi) r^2 dr \sin \theta d\theta d\phi = -i\hbar \int \phi^*(r, \theta, \phi) \frac{\partial \psi}{\partial \theta}(r, \theta, \phi) r^2 dr d\phi d\theta$$

On integrating by parts

$$= i\hbar \left[\int_0^\infty \int_0^{2\pi} (\phi^* \psi) r^2 dr d\phi + i\hbar \int_0^\infty \int_0^{2\pi} \left(\frac{\partial \phi^*}{\partial \theta} \right) \psi r^2 dr d\phi \right]$$

$$= \int_0^\infty \int_0^{2\pi} \left(\frac{+i\hbar}{\sin \theta} \frac{\partial \phi^*}{\partial \theta} \right) \psi r^2 dr \sin \theta d\theta d\phi = \int \left(\frac{-i\hbar}{\sin \theta} \frac{\partial \phi}{\partial \theta} \right)^\dagger \psi r^2 dr \sin \theta d\theta d\phi$$

$$\text{So, } \left(\frac{-i\hbar}{\sin \theta} \frac{\partial}{\partial \theta} \right)^\dagger = \frac{-i\hbar}{\sin \theta} \frac{\partial}{\partial \theta}$$

Correct answer is (c).

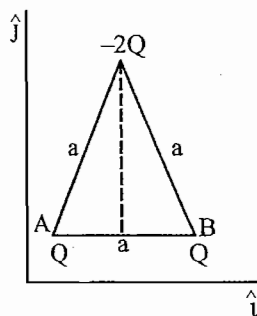
57. Which of the following quantities is Lorentz invariant?

(a) $|\mathbf{E} \times \mathbf{B}|^2$ (b) $|\mathbf{E}|^2 - |\mathbf{B}|^2$ (c) $|\mathbf{E}|^2 + |\mathbf{B}|^2$ (d) $|\mathbf{E}|^2 |\mathbf{B}|^2$

Soln. $E^2 - B^2$ is Lorentz invariant. It is a standard result of relativistic dynamics

Correct answer is (b).

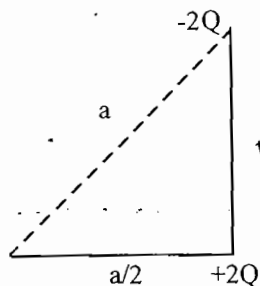
58. Charges Q , Q and $-2Q$ are placed on the vertices of an equilateral triangle ABC of sides of length a , as shown in the figure



The dipole moment of this configuration of charges, irrespective of the choice of origin, is

- (a) $+2aQ\hat{i}$ (b) $+\sqrt{3}aQ\hat{j}$ (c) $-\sqrt{3}aQ\hat{j}$ (d) 0

Soln.



$$\sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}}{2}a$$

(Distance from $-2Q$ to $+2Q$)

$$\text{Dipole moment} = (2Q) \frac{\sqrt{3}a}{2} = -\sqrt{3}aQ\hat{j}$$

Correct answer is (c).

59. The vector potential $\vec{A}$ due to a magnetic moment 'm' at a point 'r' is given by $\vec{A} = \frac{\vec{m} \times \vec{r}}{r^3}$. If $\vec{m}$ is directed along the positive z-axis, the x-component of the magnetic field, at the point r, is

- (a) $\frac{3myz}{r^5}$ (b) $\frac{3mxy}{r^5}$ (c) $\frac{3mxz}{r^5}$ (d) $\frac{3m(z^2 - xy)}{r^5}$

Soln. $\vec{\nabla} \times (\vec{u}\vec{A}) = \vec{u}\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla}\vec{u}$; $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B}\vec{\nabla})\vec{A} + \vec{A}(\vec{\nabla}\vec{B}) - (\vec{A}\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\vec{A})$

$$\vec{\nabla} \times \left[\frac{1}{r^3} (\vec{m} \times \vec{r}) \right] = \frac{1}{r^3} [\vec{\nabla} \times (\vec{m} \times \vec{r})] - (\vec{m} \times \vec{r}) \times \vec{\nabla} \frac{1}{r^3}$$

Calculate: $\vec{\nabla} \times (\vec{m} \times \vec{r})$: $\vec{m} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & m \\ x & y & z \end{vmatrix} = \hat{i}(-my) - \hat{j}(-mx)$

$$\vec{\nabla} \times (-my\hat{i} + mx\hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -my & mx & 0 \end{vmatrix} = \hat{i}.0 - \hat{j}.0 + \hat{k}(m+m) = 2m\hat{k}$$

The x-component of this term = zero

$$\vec{\nabla} \frac{1}{r^3} = \hat{i} \frac{\partial}{\partial x} \frac{1}{r^3} + \hat{j} \frac{\partial}{\partial y} \frac{1}{r^3} + \hat{k} \frac{\partial}{\partial z} \frac{1}{r^3}; \frac{1}{r^3} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) = -\frac{3}{2} \frac{2x}{(x^2 + y^2 + z^2)^{5/2}} = -\frac{3x}{r^5} \Rightarrow \vec{\nabla} \frac{1}{r^3} = -3 \frac{[\hat{i}x + \hat{j}y + \hat{k}z]}{r^5} = -\frac{3\vec{r}}{r^5}$$

$$\begin{aligned} \therefore (\vec{m} \times \vec{r}) \times \vec{\nabla} \frac{1}{r^3} &= (-my\hat{i} + mx\hat{j}) \times \left[\hat{i}x + \hat{j}y + \hat{k}z \right] \left(\frac{-3}{r^5} \right) = +\frac{3m}{r^5} [(y\hat{i} - x\hat{j}) \times (\hat{i}x + \hat{j}y + \hat{k}z)] \\ &= \frac{3m}{r^5} [xy(\hat{i} \times \hat{i}) + y^2(\hat{i} \times \hat{j}) + yz(\hat{i} \times \hat{k}) - x^2(\hat{j} \times \hat{i}) + xy(\hat{j} \times \hat{j}) + xz(\hat{j} \times \hat{k})] \\ &= \frac{3m}{r^5} [(y^2 + x^2)\hat{k} - yz\hat{j} + xz\hat{i}] \end{aligned}$$

The x -component $= \frac{3m}{r^5} xz$

Correct answer is (c).

60. A system has two normal modes of vibration, with frequencies ω_1 and $\omega_2 = 2\omega_1$. What is the probability that at temperature T, the system has an energy less than $4\hbar\omega_1$? [In the following $x = e^{-\beta\hbar\omega}$ and Z is the partition function]

- (a) $x^{3/2} (x + 2x^2) / Z$ (b) $x^{3/2} (1 + x + x^2) / Z$
(c) $x^{3/2} (1 + 2x^2) / Z$ (d) $x^{3/2} (1 + x + 2x^2) / Z$

Soln. The possible energies of the system are given by

$$= \left(n + \frac{1}{2}\right)\hbar\omega_1 + \left(m + \frac{1}{2}\right)\hbar\omega_2, \text{ with } n, m = 0, 1, 2, 3, \dots = \left(n + \frac{1}{2}\right)\hbar\omega_1 + \left(m + \frac{1}{2}\right)2\hbar\omega_1$$

The possible energies with corresponding degeneracies (that are less than $4\hbar\omega_1$)

$$E_1 = \frac{1}{2}\hbar\omega_1 + \frac{1}{2} \times 2\hbar\omega_1 \quad (n=0, m=0); \quad E_1 = \frac{3}{2}\hbar\omega_1 \quad (g_1=1)$$

$$E_2 = \frac{3}{2}\hbar\omega_1 + \frac{1}{2} \times 2\hbar\omega_1 \quad (n=1, m=0); \quad E_2 = \frac{5}{2}\hbar\omega_1 \quad (g_2=1)$$

$$E_3 = \frac{1}{2}\hbar\omega_1 + \frac{3}{2} \times 2\hbar\omega_1 = \frac{7}{2}\hbar\omega_1 \quad (n=0, m=1)$$

$$E_3 = \frac{5}{2}\hbar\omega_1 + \frac{1}{2} \times 2\hbar\omega_1 = \frac{7}{2}\hbar\omega_1 \quad (n=2, m=0); \quad E_3 = \frac{7}{2}\hbar\omega_1 \quad (g_3=2)$$

The probability that at temperature T, the system has energy less than $4\hbar\omega_1$

$$p = p_1 + p_2 + p_3 \quad \left(\text{where, } p_i = \frac{g_i \exp(-\beta E_i)}{Z} \right)$$

$$= \frac{e^{-\frac{3}{2}\beta\hbar\omega_1}}{Z} + \frac{e^{-\frac{5}{2}\beta\hbar\omega_1}}{Z} + \frac{2e^{-\frac{7}{2}\beta\hbar\omega_1}}{Z} = \frac{e^{-\frac{3}{2}\beta\hbar\omega_1}}{Z} (1 + e^{-\beta\hbar\omega_1} + 2e^{-2\beta\hbar\omega_1}) = \frac{x^{3/2}}{Z} (1 + x + x^2)$$

Correct answer is (d).

61. The magnetization M of a ferromagnet, as a function of the temperature T and the magnetic field H, is described by the equation $M = \tanh \left(\frac{T_c}{T} M + \frac{H}{T} \right)$. In these units, the zero-field magnetic susceptibility in terms of $M(0) = M(H=0)$ is given by

- (a) $\frac{1-M^2(0)}{T-T_c(1-M^2(0))}$ (b) $\frac{1-M^2(0)}{T-T_c}$ (c) $\frac{1-M^2(0)}{T+T_c}$ (d) $\frac{1-M^2(0)}{T}$

Soln. $M = \tanh \left(\frac{T_c}{T} M + \frac{H}{T} \right) \Rightarrow \frac{\partial M}{\partial H} = \left[1 - \tanh^2 \left(\frac{T_c}{T} M + \frac{H}{T} \right) \right] \left(\frac{T_c}{T} \frac{\partial M}{\partial H} + \frac{1}{T} \right)$

$$\Rightarrow \frac{\partial M}{\partial H} = \frac{(1-M^2)}{T} \left(1 + T_c \frac{\partial M}{\partial H} \right) = \frac{1-M^2}{T} + (1-M^2) \left(\frac{T_c}{T} \right) \frac{\partial M}{\partial H}$$

$$\Rightarrow \frac{\partial M}{\partial H} \left[1 - (1-M^2) \frac{T_c}{T} \right] = \frac{1-M^2}{T} \Rightarrow \frac{\partial M}{\partial H} \left[\frac{T - T_c(1-M^2)}{T} \right] = \frac{1-M^2}{T}$$

$$\Rightarrow \left. \frac{\partial M}{\partial H} \right|_{H=0} = \frac{1-M^2}{T - T_c(1-M^2)} = \frac{1-M(0)^2}{T - T_c(1-M^2(0))}$$

Correct answer is (a).

62. Bose condensation occurs in liquid He^4 kept at ambient pressure at 2.17 K. At which temperature will Bose condensation occur in He^4 in gaseous state, the density of which is 1000 times smaller than that of liquid He^4 ? (Assume that it is a perfect Bose gas)

(a) 2.17 mK (b) 21.7 mK (c) 21.7 μK (d) 2.17 μK

Soln. Bose condensation Temperature $T_B \propto n^{-2/3}$, n = particle density

$$\frac{(T_B)_{\text{liquid}}}{(T_B)_{\text{gas}}} = \left(\frac{n_{\text{liquid}}}{n_{\text{gas}}} \right)^{2/3} = \left(\frac{n_{\text{liquid}}}{\frac{n_{\text{liquid}}}{1000}} \right)^{2/3} = (1000)^{2/3} = (10)^3 = 100$$

$$\Rightarrow (T_B)_{\text{gas}} = \frac{(T_B)_{\text{liquid}}}{100} = \frac{2.17\text{K}}{100} = 21.7\text{mK}$$

Correct answer is (b).

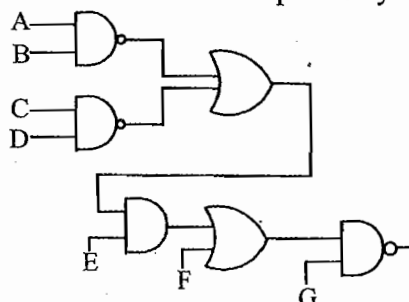
63. Consider black body radiation contained in a cavity whose walls are at temperature T . The radiation is in equilibrium with the walls of the cavity. If the temperature of the walls is increased to $2T$ and the radiation is allowed to come to equilibrium at the new temperature, the entropy of the radiation increases by a factor of

(a) 2 (b) 4 (c) 8 (d) 16

Soln. In black body radiation, entropy, $S \propto T^3 \Rightarrow \frac{S_1}{S_2} = \left(\frac{T_1}{T_2} \right)^3 \Rightarrow S_2 = \left(\frac{2T}{T} \right)^3 S_1 = 8S_1$

Correct answer is (c).

64. The output 0, of the given circuit in cases I and II, where
Case I : A, B = 1; C, D = 0; E, F = 1 and G = 0
Case II : A, B = 0; C, D = 0; E, F = 0 and G = 1 are respectively



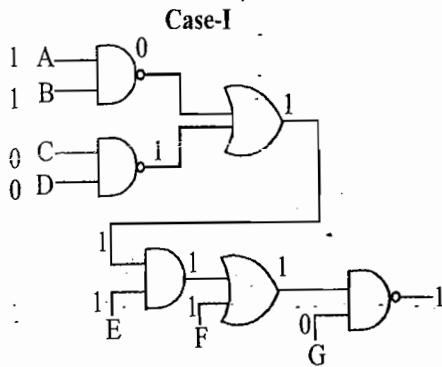
(a) 1, 0

(b) 0, 1

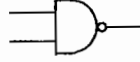
(c) 0, 0

(d) 1, 1

Soln.

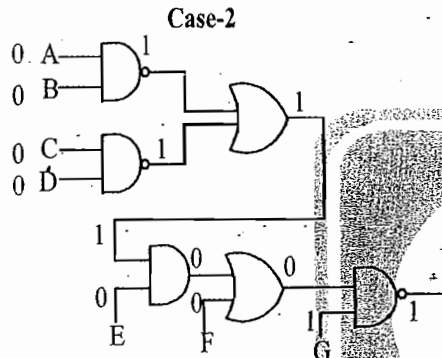


NAND



Truth Table

A	B	output
0	0	1
0	1	1
1	0	1
1	1	0



A	B	output
0	0	0
0	1	1
1	0	1
1	1	1

A	B	output
0	0	0
0	1	0
1	0	0
1	1	1

Output is (1)

Correct answer is (d).

CAREER ENDEAVOUR

65. A resistance strain gauge is fastened to a steel fixture and subjected to a stress of 1000 kg/m^2 . If the gauge factor is 3 and the modulus of elasticity of steel is $2 \times 10^{10} \text{ kg/m}^2$, then the fractional change in resistance of the strain gauge due to the applied stress is:

(Note: The gauge factor is defined as the ratio of the fractional change in resistance to the fractional change in length.)

(a) 1.5×10^{-7}

(b) 3.0×10^{-7}

(c) 0.16×10^{-10}

(d) 0.5×10^{-7}

Soln. $\frac{\text{Stress}}{\text{Strain}} = \text{Young modulus of elasticity} \Rightarrow \frac{1000 \text{ kg/m}^2}{\text{strain}} = 2 \times 10^{10} \text{ kg/m}^2$

$$\Rightarrow \text{Strain} = \frac{10^3}{2 \times 10^{10}} = 0.5 \times 10^{-7}$$

Fractional change in resistance.

Correct answer is (a).

66. Consider a sinusoidal waveform of amplitude 1V and frequency f_0 . Starting from an arbitrary initial time, the waveform is sampled at intervals of $1/(2f_0)$. If the corresponding Fourier spectrum peaks at a frequency $\bar{f}$ and an amplitude $\bar{A}$, then

- (a) $\bar{f} = 2f_0$ and $\bar{A} = 1V$ (b) $\bar{f} = f_0$ and $0 \leq \bar{A} \leq 1V$
 (c) $\bar{f} = 0$ and $\bar{A} = 1V$ (d) $\bar{f} = \frac{f_0}{2}$ and $\bar{A} = \frac{1}{\sqrt{2}}V$

Soln. $\bar{f}$ is the Nyquist frequency which is always $\frac{1}{2}$ of sampling rate, $\bar{f} = \frac{1}{2}f_0$

$$\bar{A} = \text{rms value of sine wave} = \frac{1}{\sqrt{2}}V$$

$$\left(\overline{\sin^2 \omega t} + \overline{\cos^2 \omega t} = 1; \overline{\sin^2 \omega t} = \frac{1}{2}; \left(\overline{\sin^2 \omega t} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \right)$$

Correct answer is (d).

67. The first absorption spectrum of $^{12}\text{C}^{16}\text{O}$ is at 3.842cm^{-1} while that of $^{13}\text{C}^{16}\text{O}$ is at 3.673cm^{-1} . The ratio of their moments of inertia is
 (a) 1.851 (b) 1.286 (c) 1.046 (d) 1.038

Soln. First absorption spectra corresponds to $J=0 \rightarrow J=1$ and corresponding frequency is $2B$.

$$\text{Now, } \frac{(B)_{^{12}\text{C}^{16}\text{O}}}{(B)_{^{13}\text{C}^{16}\text{O}}} = \frac{3.845}{3.627} = \frac{(I)_{^{13}\text{C}^{16}\text{O}}}{(I)_{^{12}\text{C}^{16}\text{O}}} = 1.046$$

Correct answer is (c).

68. The spin-orbit interaction in an atom is given by $H = a \vec{L} \cdot \vec{S}$, where L and S denote the orbital and spin angular momentum, respectively of electron. The splitting between the levels $^2P_{3/2}$ and $^2P_{1/2}$ is:

- (a) $\frac{3}{2}a\hbar^2$ (b) $\frac{1}{2}a\hbar^2$ (c) $3a\hbar^2$ (d) $\frac{5}{2}a\hbar^2$

Soln. $H = a \vec{L} \cdot \vec{S} = a \frac{[J^2 - L^2 - S^2]}{2} = \frac{a}{2} [j(j+1)\hbar^2 - \ell(\ell+1)\hbar^2 - s(s+1)\hbar^2]$ $l=1, s=1/2$

$$\Delta E = E_{j=3/2} - E_{j=1/2} = \frac{a}{2} \left[\frac{3}{2} \times \frac{5}{2} \hbar^2 - \frac{1}{2} \times \frac{3}{2} \hbar^2 \right] = \frac{a\hbar^2}{2} \left[\frac{15}{4} - \frac{3}{4} \right] = \frac{3}{2} a\hbar^2$$

Correct answer is (a).

69. The spectral line corresponding to an atomic transition from $J=1$ to $J=0$ states splits in a magnetic field of 1 KG into three components separated by $1.6 \times 10^{-3} \text{Å}$. If the zero field spectral line corresponding to 1849Å , what is the g-factor corresponding to the $J=1$ state? (You may use

$$\frac{hc}{\mu_0} \approx 2 \times 10^4 \text{ cm.})$$

- (a) 2 (b) 3/2 (c) 1 (d) 1/2

Soln. $\Delta E = g\mu_B B \Rightarrow \Delta(h\nu) = \Delta\left(\frac{hc}{\lambda}\right) = -hc \frac{\Delta\lambda}{\lambda^2} = hc \frac{|\Delta\lambda|}{\lambda^2} = g\mu_B B$

$$\Rightarrow g = \left(\frac{hc}{\mu_B}\right) \frac{\Delta\lambda}{\lambda^2 B} = \frac{(2.14 \times 10^4 \text{ cm.G})(1.6 \times 10^{-11} \text{ cm})}{(1.849 \times 10^{-5}) \times 10^3} = 1$$

Correct answer is (c).

70. The energy required to create a lattice vacancy in a crystal is equal to 1 eV. The ratio of the number densities of vacancies $n(1200\text{K})/n(300\text{K})$, when the crystal is at equilibrium at 1200K and 300K, respectively is approximately

(a) $\exp(-30)$ (b) $\exp(-15)$ (c) $\exp(15)$ (d) $\exp(30)$

Soln. Use Boltzman factor, ($1\text{eV} = 11605\text{K}$)

$$\frac{n(1200\text{K})}{n(300\text{K})} = \frac{e^{-\frac{11605}{1200}}}{e^{-\frac{11605}{300}}} = e^{-\frac{11605}{1200} + \frac{11605}{300}} = e^{-\frac{11605}{300} \left(1 - \frac{1}{4}\right)} = e^{-\frac{11605 \times 3}{400}} \sim e^{-\frac{11605}{133}} \sim e^{-30}$$

Correct answer is (d).

71. The dispersion relation of phonons in a solid is given by

$$\omega^2(k) = \omega_0^2 (3 - \cos k_x a - \cos k_y a - \cos k_z a)$$

The velocity of the phonons at large wavelength is

(a) $\omega_0 a / \sqrt{3}$ (b) $\omega_0 a$ (c) $\sqrt{3} \omega_0 a$ (d) $\omega_0 a / \sqrt{2}$

Soln. $\omega^2(k) = \omega_0^2 (3 - \cos k_x a - \cos k_y a - \cos k_z a)$

Expand $\cos \theta = 1 - \frac{\theta^2}{2!}$ (For θ is small)

For larger λ , ($k_x a, k_y a, k_z a$) are small.

$$\omega^2(k) = \omega_0^2 \left[3 - \left(1 - \frac{k_x^2 a^2}{2}\right) - \left(1 - \frac{k_y^2 a^2}{2}\right) - \left(1 - \frac{k_z^2 a^2}{2}\right) \right] = \frac{\omega_0^2}{2} a^2 (k_x^2 + k_y^2 + k_z^2) = \frac{\omega_0^2 a^2 k^2}{2}$$

$$\Rightarrow \omega(k) = \frac{\omega_0 a}{\sqrt{2}} k \Rightarrow \text{velocity} = \frac{d\omega}{dk} = \frac{\omega_0 a}{\sqrt{2}}$$

Correct answer is (d).

72. Consider an electron in a box of length L with periodic boundary condition $\psi(x) = \psi(x+L)$. If the

electron is in the $\psi_k(x) = \frac{1}{\sqrt{L}} e^{ikx}$ with energy $\epsilon_k = \frac{\hbar^2 k^2}{2m}$, what is the correction to its energy, to second order of perturbation theory, when it is subjected to weak periodic potential $V(x) = V_0 \cos gx$, where g is an integral multiple of the $2\pi/L$?

(a) $V_0^2 \epsilon_g / \epsilon_k^2$

(b) $-\frac{mV_0^2}{2\hbar^2} \left(\frac{1}{g^2 + 2kg} + \frac{1}{g^2 - 2kg} \right)$

(c) $V_0^2 (\epsilon_k - \epsilon_g) / \epsilon_g^2$

(d) $V_0^2 / (\epsilon_k + \epsilon_g)$

Soln. $\psi(x) = \psi(x+L) \Rightarrow \frac{1}{\sqrt{L}} e^{ikx} = \frac{1}{\sqrt{L}} e^{ik(x+L)} \Rightarrow \boxed{e^{ikL} = 1} \Rightarrow kL = 2n\pi \Rightarrow k = \frac{2n\pi}{L}$

Perturbed periodic potential, $V(x) = V_0 \cos gx$, $g = \frac{2m\pi}{L}$, $m = 1, 2, 3, 4, \dots$

Second order correction to energy, $\epsilon_K^{(2)} = \sum_{K \neq K'} \frac{|\langle \psi_{K'} | V(x) | \psi_K \rangle|^2}{\epsilon_K - \epsilon_{K'}}$

$$\begin{aligned} \langle \psi_{K'} | V(x) | \psi_K \rangle &= \int_{-\infty}^{+\infty} \psi_{K'}^* V_0 \cos gx \psi_K dx / \int \psi_{K'}^* \psi_K dx \\ &= \frac{V_0}{L} \int_0^L e^{-ik'x} \frac{(e^{igx} + e^{-igx})}{2} e^{ikx} dx = \frac{V_0}{2} \left[\int_0^L e^{igx} e^{i(k-k')x} dx + \int_0^L e^{-igx} e^{i(k-k')x} dx \right] \\ &= \frac{1}{L} \int_0^L e^{-ik'x} e^{ikx} dx = \int_0^L e^{i(k-k')x} dx \\ \Rightarrow \epsilon_K^{(2)} &= \sum_{K' \neq K} \frac{|\langle \psi_{K'} | V(x) | \psi_K \rangle|^2}{\epsilon_K - \epsilon_{K'}} \end{aligned}$$

Correct answer is (b).

73. The ground state of $^{207}_{82}\text{Pb}$ nucleus has spin-parity $J^P = \frac{1}{2}^-$, while the first excited state has $J^P = \frac{5}{2}^-$. The electromagnetic radiation emitted when the nucleus makes a transition from the first excited state to the ground state are
 (a) E2 and E3 (b) M2 and E3 (c) E2 and M3 (d) M2 and M3

Soln. No parity change; $\Delta J = 2, 3$

For E_ℓ type, $\Delta\pi = (-1)^\ell$, (for no parity change $\ell = 2$)

For M_ℓ type, $\Delta\pi = (-1)^{\ell+1}$, (for no parity change $\ell = 3$)

$\Delta J = 2$, No parity change $\rightarrow E2$; $\Delta J = 3$, No parity change $\rightarrow M3$

Correct answer is (c)

74. The dominant interactions underlying the following processes

A. $K^- + p \rightarrow \Sigma^- + \pi^+$, B. $\mu^- + \mu^+ \rightarrow K^- + K^+$, C. $\Sigma^+ \rightarrow p + \pi^0$ are

(a) A: strong, B: electromagnetic and C: weak

(b) A: strong, B: weak and C:

(c) A: weak, B: electromagnetic and C: strong and C: weak

(d) A: weak, B: electromagnetic

Soln. (C) $\Sigma^+ \rightarrow p + \pi^0$

$I_3: 1 \rightarrow \frac{1}{2} + 0$ (Not conserved), I_3 is the Z component of isotopic spin

Weak interaction

(B) This reaction involves μ^+ and μ^- which are leptons which are charged

Electromagnetic interaction

(A) $K^- + p \rightarrow \Sigma^- + \pi^+$ involves hadrons; $I_3 : -\frac{1}{2} + \frac{1}{2} \rightarrow -1 + 1$ (Conserved)

Strong interaction.

Correct answer is (a).

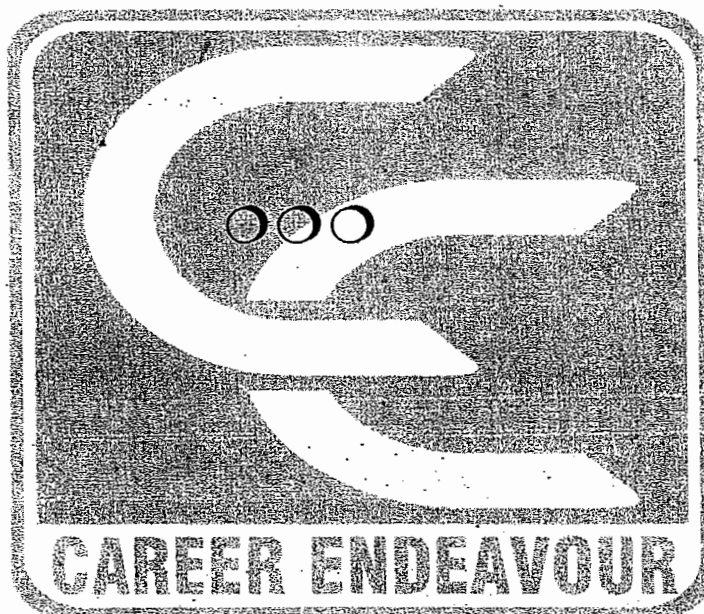
75. If a Higgs boson of mass m_H with a speed $\beta = \frac{v}{c}$ decays into a pair photons, then the invariant mass of the photon pair is

[Note: The invariant mass of a system of two particles, with four-momenta p_1 and p_2 is $(p_1 + p_2)^2$]

- (a) βm_H (b) m_H (c) $m_H / \sqrt{1 - \beta^2}$ (d) $\beta m_H / \sqrt{1 - \beta^2}$

Soln. The mass of photon pair = mass of the initial particle = m_H

Correct answer is (b).



CSIR-UGC-NET/JRF- DEC. - 2012
PHYSICAL SCIENCES BOOKLET - [A]

Part-B

21. A 2×2 matrix A has eigenvalues $e^{i\pi/5}$ and $e^{i\pi/6}$. The smallest value of 'n' such that $A^n = I$ is:
(a) 20 (b) 30 (c) 60 (d) 120

Soln. A has eigenvalues $e^{i\frac{\pi}{5}}$ and $e^{i\frac{\pi}{6}}$

A^n has eigenvalues $e^{i\frac{n\pi}{5}}$ and $e^{i\frac{n\pi}{6}}$

So, $|A^n| = \text{product of eigenvalues} = e^{i\frac{11n\pi}{30}}$

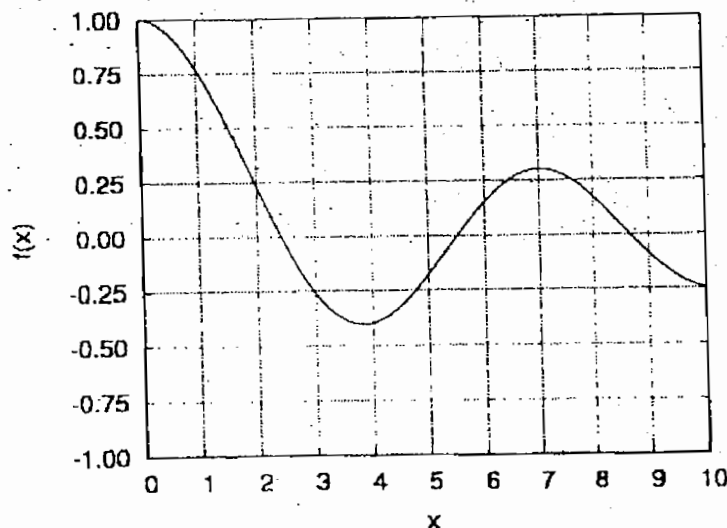
Since, $A^n = I \Rightarrow |A^n| = 1$

$$e^{i(11n\pi/30)} = 1 = e^{2m\pi i} \Rightarrow \frac{11n\pi}{30} = 2m\pi \Rightarrow \frac{n}{m} = \frac{60}{11}$$

So, the smallest value of 'n' is 60.

Correct option is (c)

22. The graph of the function $f(x)$ as shown below is best described by



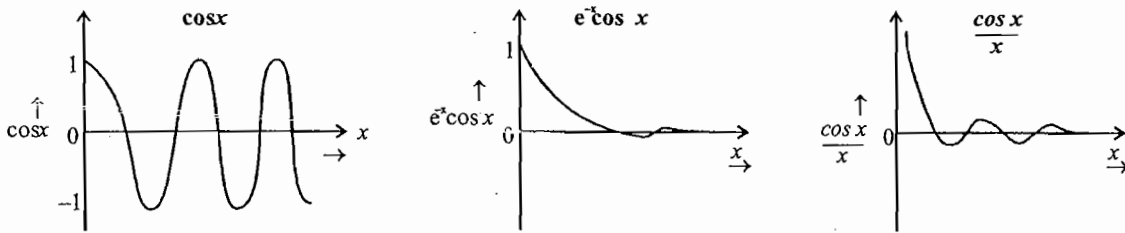
- (a) The Bessel function $J_0(x)$ (b) $\cos x$
(c) $e^{-x} \cos x$ (d) $\frac{1}{x} \cos x$

Soln. At $x=0$, Bessel function $J_0(x) = 1$

$J_0(x)$ has zeroes at $x = 2.4048, 5.5201, 8.6537$ etc.

So, the graph is best described by $J_0(x)$

The other functions are following plots.



Correct option is (a).

23. In a series of five cricket matches, one of the captains calls "Heads" every time when the toss is taken. The probability that he will win 3 times and lose 2 times is
 (a) $1/8$ (b) $5/8$ (c) $3/16$ (d) $5/16$

Soln. Number of trials i.e. tosses 'n' = 5

Number of success i.e. getting heads 'r' = 3

Probability of success i.e. getting head in one trial 'p' = $1/2$

Probability of failure i.e. getting tail in one trial 'q' = $1/2$

Required probability: $P = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{5!}{3!2!} \times \frac{1}{32} = \frac{5 \cdot 4}{2} \times \frac{1}{32} = \frac{5}{16}$

Correct option is (d)

24. The unit normal vector at the point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ on the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ is}$$

- (a) $\frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{a^2 + b^2 + c^2}}$ (b) $\frac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2 + b^2 + c^2}}$ (c) $\frac{b\hat{i} + c\hat{j} + a\hat{k}}{\sqrt{a^2 + b^2 + c^2}}$ (d) $\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

Soln. Given surface of ellipsoid: $\phi: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Normal to the surface at point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$:

$$\vec{\nabla} \phi \Big|_{\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)} = \frac{2x}{a^2} \hat{i} + \frac{2y}{b^2} \hat{j} + \frac{2z}{c^2} \hat{k} = \frac{2}{\sqrt{3}a} \hat{i} + \frac{2}{\sqrt{3}b} \hat{j} + \frac{2}{\sqrt{3}c} \hat{k}$$

Unit normal vector to the surface at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$:

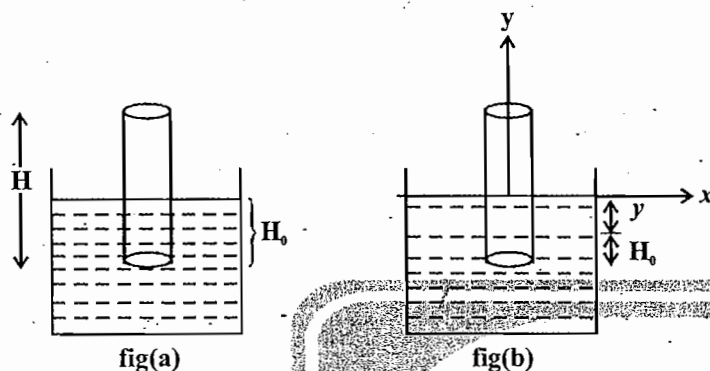
$$\hat{n} = \frac{\vec{\nabla} \phi}{|\vec{\nabla} \phi|} = \frac{\frac{2}{\sqrt{3}} \left(\frac{1}{a} \hat{i} + \frac{1}{b} \hat{j} + \frac{1}{c} \hat{k} \right)}{\frac{2}{\sqrt{3}} \sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)}} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$

Option (a) is correct but it is wrongly printed in the question.

25. A solid cylinder of height H , radius R and density ρ , floats vertically on the surface of a liquid of density ρ_0 . The cylinder will be set into oscillatory motion when a small instantaneous downward force is applied. The frequency of oscillation is

(a) $\frac{\rho g}{\rho_0 H}$ (b) $\frac{\rho}{\rho_0} \sqrt{\frac{g}{H}}$ (c) $\sqrt{\frac{\rho g}{\rho_0 H}}$ (d) $\sqrt{\frac{\rho_0 g}{\rho H}}$

Soln.



(a) Cylinder is partially submerged and in equilibrium

(b) Cylinder is not in equilibrium since the force of buoyancy on cylinder is not equal to weight of the cylinder.

$$AH\rho g = AH_0\rho_0 g \quad (\text{for equilibrium}) \quad (A = \pi R^2)$$

For a displacement y , the equation of motion is

$$\Rightarrow AH\rho \frac{d^2 y}{dt^2} = -AH\rho g + A(H_0 - y)\rho_0 g$$

$$\Rightarrow \frac{d^2 y}{dt^2} = -g + \frac{(H_0 - y)\rho_0 g}{H\rho} = -g + \frac{H_0\rho_0 g}{H\rho} - \frac{y\rho_0 g}{H\rho}$$

Since $H_0\rho_0 = H\rho$

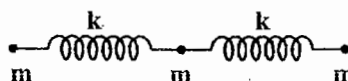
$$\Rightarrow \frac{d^2 y}{dt^2} = -g + g - \frac{\rho_0 g}{\rho H} y \Rightarrow \frac{d^2 y}{dt^2} = -\left(\frac{\rho_0 g}{\rho H}\right) y$$

Acceleration is proportional to displacement, motion is simple harmonic

$$\Rightarrow \omega^2 = \frac{\rho_0 g}{\rho H} \Rightarrow \omega = \sqrt{\frac{\rho_0 g}{\rho H}}$$

Correct option is (d)

26. Three particles of equal mass ' m ' are connected by two identical massless springs of stiffness constant ' k ' as shown in the figure.



If x_1 , x_2 and x_3 denote the displacements of the masses from their respective equilibrium positions, the potential energy of the system is:

(a) $\frac{1}{2}k(x_1^2 + x_2^2 + x_3^2)$

(b) $\frac{1}{2}k[x_1^2 + x_2^2 + x_3^2 - x_2(x_1 + x_3)]$

(c) $\frac{1}{2}k[x_1^2 + 2x_2^2 + x_3^2 + 2x_2(x_1 + x_3)]$

(d) $\frac{1}{2}k[x_1^2 + 2x_2^2 + x_3^2 - 2x_2(x_1 + x_3)]$

Soln. Potential energy = $\frac{1}{2}k(x_2 - x_1)^2$ first spring

Potential energy = $\frac{1}{2}k(x_3 - x_2)^2$ second spring

Potential energy of the system,

$$= \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2$$

$$= \frac{1}{2}k[x_1^2 + x_3^2 - 2x_1x_2 + x_2^2 + x_3^2 - 2x_2x_3]$$

$$= \frac{1}{2}k[x_1^2 + 2x_2^2 + x_3^2 - 2x_2(x_1 + x_3)]$$

Correct option is (d)

27. Let v , p and E denotes the speed, the magnitude of the momentum, and the energy of a free particle of rest mass ' m '. Then

(a) $\frac{dE}{dp} = \text{constant}$

(b) $p = mv$

(c) $v = \frac{cp}{\sqrt{p^2 + m^2c^2}}$

(d) $E = mc^2$

Soln. The energy of a particle of rest mass ' m ' is given by $E = \sqrt{c^2p^2 + m^2c^4}$

The velocity of particle,

$$v = v_g = \text{group velocity}$$

$$= \frac{dE}{dp} = \frac{\frac{1}{2} \times 2c^2p}{\sqrt{c^2p^2 + m^2c^4}} = \frac{c^2p}{c\sqrt{p^2 + m^2c^2}} = \frac{cp}{\sqrt{p^2 + m^2c^2}}$$

Correct option is (c).

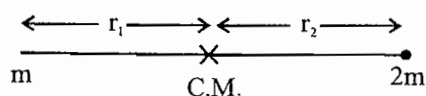
28. A binary star system consists of two stars S_1 and S_2 , with masses m and $2m$ respectively separated by a distance ' r '. If both S_1 and S_2 individually follow circular orbits around the centre of the mass with instantaneous speeds v_1 and v_2 respectively, the ratio of speeds v_1/v_2 is:

(a) $\sqrt{2}$

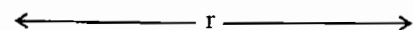
(b) 1

(c) $1/2$

(d) 2



Soln.



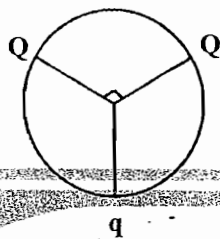
Let centre of mass C.M. lie at a distance r_1 from m and r_2 from $2m$, then by definition of C.M.

$$mr_1 = 2mr_2$$

$$\Rightarrow m \frac{dr_1}{dt} = 2m \frac{dr_2}{dt} \Rightarrow mv_1 = 2mv_2 \Rightarrow \frac{v_1}{v_2} = \frac{2m}{m} = 2 \Rightarrow \frac{v_1}{v_2} = 2$$

Correct option is (d)

29. Three charges are located on the circumference of a circle of radius 'R' as shown in the figure below. The two charges Q subtend an angle 90° at the centre of the circle. The charge ' q ' is symmetrically placed with respect to the charges Q . If the electric field at the centre of the circle is zero, what is the magnitude of Q ?



(a) $q/\sqrt{2}$

(b) $\sqrt{2}q$

(c) $2q$

(d) $4q$

Soln. Let radius of circle is R .

Electric field due to charge ' q ' at centre

$$E_1 = \frac{q}{4\pi\epsilon_0 R^2} \quad (\text{up direction}) \quad (1)$$

Electric field due to both Q charges is

$$\vec{E}_2 = \vec{E}_3 = \frac{Q}{4\pi\epsilon_0 R^2} \quad (\text{aligned } 45^\circ \text{ from downward direction})$$

The resultant electric field of two Q charges will be in downward directions.

Hence,
$$\frac{2Q}{4\pi\epsilon_0 R^2} \sin 45^\circ = \frac{2Q}{4\pi\epsilon_0 R^2} \frac{1}{\sqrt{2}} \quad \dots (2)$$

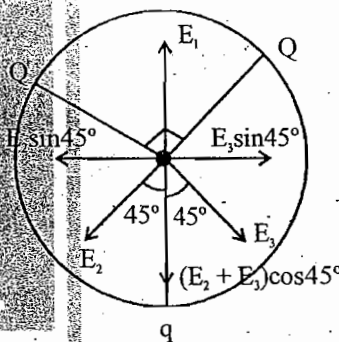
From (1) and (2), electric fields must be equal

$$\frac{q}{4\pi\epsilon_0 R^2} = \frac{2Q}{4\pi\epsilon_0 R^2} \frac{1}{\sqrt{2}}$$

$$Q = \frac{q}{\sqrt{2}}$$

Correct option is (a)

30. Consider a hollow charged shell of inner radius ' a ' and outer radius ' b '. The volume charge density is $\rho(r) = \frac{k}{r^2}$ (where k is a constant) in the region $a < r < b$. The magnitude of the electric field produced at distance $r > a$ is:



- (a) $\frac{k(b-a)}{\epsilon_0 r^2}$ for $r > a$
- (b) $\frac{k(b-a)}{\epsilon_0 r^2}$ for $a < r < b$ and $\frac{kb}{\epsilon_0 r^2}$ for $r > b$
- (c) $\frac{k(r-a)}{\epsilon_0 r^2}$ for $a < r < b$ and $\frac{k(b-a)}{\epsilon_0 r^2}$ for $r > b$
- (d) $\frac{k(r-a)}{\epsilon_0 a^2}$ for $a < r < b$ and $\frac{k(b-a)}{\epsilon_0 a^2}$ for $r > b$

Soln.

$$\rho = \frac{k}{r^2}$$

$$a < r < b$$

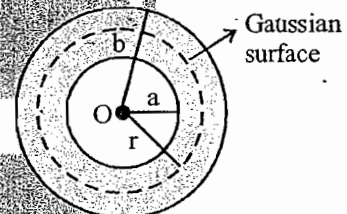
Electric field for $a < r < b$:

Charge enclosed by Gaussian surface

$$q_{\text{enc}} = \int_a^r \frac{k}{r^2} \cdot 4\pi r^2 dr = 4\pi k(r-a)$$

From Gauss law, $\oint_S \vec{E} \cdot d\vec{S} = \frac{q_{\text{enc}}}{\epsilon_0} \Rightarrow E \cdot 4\pi r^2 = \frac{4\pi k(r-a)}{\epsilon_0}$

$$\vec{E} = \frac{k(r-a)}{\epsilon_0 r^2} \hat{r} \quad \text{for } a < r < b$$



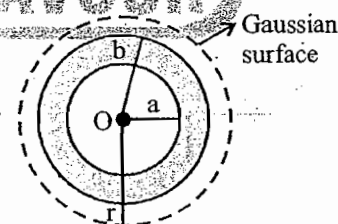
Electric field for $r > b$:

Now charge enclosed by Gaussian surface

$$q_{\text{enc}} = \int_a^b \frac{k}{r^2} \cdot 4\pi r^2 dr = 4\pi(b-a)k$$

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{q_{\text{enc}}}{\epsilon_0} \Rightarrow E \cdot 4\pi r^2 = \frac{4\pi(b-a)k}{\epsilon_0}$$

$$\vec{E} = \frac{k(b-a)}{\epsilon_0 r^2} \hat{r} \quad r > b$$



Correct option is (c)

31. Consider the interference of two coherent electromagnetic waves whose electric field vectors are given by $\vec{E}_1 = \hat{i}E_0 \cos \omega t$ and $\vec{E}_2 = \hat{j}E_0 \cos(\omega t + \phi)$ where ϕ is the phase difference. The intensity of the resulting wave is given by $\frac{\epsilon_0}{2} \langle E^2 \rangle$, where $\langle E^2 \rangle$ is the time average of E^2 . The total intensity is

- (a) 0 (b) $\epsilon_0 E_0^2$ (c) $\epsilon_0 E_0^2 \sin^2 \phi$ (d) $\epsilon_0 E_0^2 \cos^2 \phi$

Soln. $\vec{E}_1 = \hat{i}E_0 \cos \omega t$, $\vec{E}_2 = \hat{j}E_0 \cos(\omega t + \phi)$, $\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2}$

Intensity, $I = \frac{\epsilon_0}{2} \langle E^2 \rangle$

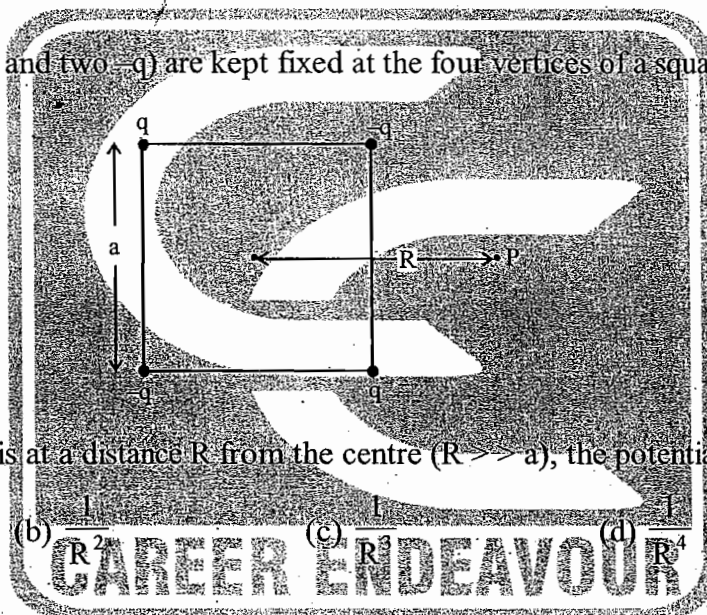
$\vec{E} = \vec{E}_1 + \vec{E}_2 \Rightarrow \boxed{\langle E^2 \rangle = 2E_0^2}$

No effect of phase (ϕ) on average value of total electric field ($\vec{E}$) as electric fields are perpendicular to each other.

Hence, $\boxed{I = \frac{\epsilon_0}{2} 2E_0^2 = \epsilon_0 E_0^2}$

Correct option is (b)

32. Four charges (two $+q$ and two $-q$) are kept fixed at the four vertices of a square of side 'a' as shown



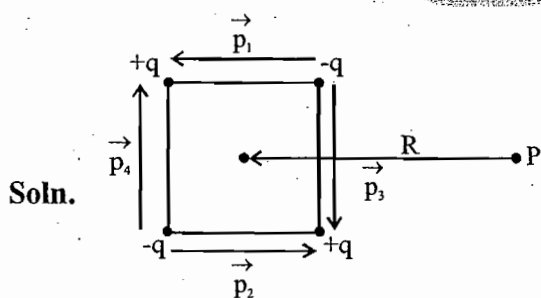
At the point P which is at a distance R from the centre ($R \gg a$), the potential is proportional to

(a) $\frac{1}{R}$

(b) $\frac{1}{R^2}$

(c) $\frac{1}{R^3}$

(d) $\frac{1}{R^4}$



Monopole moment of this charge configuration $= +q - q + q - q = 0$

Dipole moment $= \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4 = 0$

But quadrupole moment $\neq 0$

hence, V for ($R \gg a$), $\boxed{V \propto \frac{1}{R^3}}$

Correct option is (c)

33. A point charge 'q' of mass 'm' is kept at a distance 'd' below a grounded infinite conducting sheet which lies in the xy-plane. What is the value of 'd' for which the charge remains stationary?

- (a) $q/4\sqrt{mg\pi\epsilon_0}$ (b) $q/\sqrt{mg\pi\epsilon_0}$
(c) There is no finite value of 'd' (d) $\sqrt{mg\pi\epsilon_0}/q$

Soln. Force on a point charge 'q' due to infinite grounding conducting plane is given by

$$\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d^2} \quad (\text{upward direction}) \quad \dots (1)$$

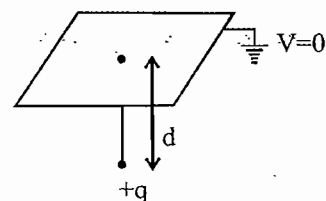
While its weight = mg (downward direction) ... (2)

Upward coulomb force in equation (1) balances the downward weight in equation (2) hence by (1) and (2), we get

$$\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d^2} = mg$$

$$\Rightarrow d = \frac{q}{4\sqrt{\pi\epsilon_0 mg}}$$

Correct option is (a).



34. The wave function of a state of the hydrogen atom is given by

$$\psi = \psi_{200} + 2\psi_{211} + 3\psi_{210} + \sqrt{2}\psi_{21-1}$$

where ψ_{nlm} denotes the normalized eigen function of the state with quantum numbers n , l and m in the usual notation. The expectation value of L_z in the state ψ is

- (a) $\frac{15\hbar}{16}$ (b) $\frac{11\hbar}{16}$ (c) $\frac{3\hbar}{8}$ (d) $\frac{\hbar}{8}$

Soln. The given wavefunction is $\psi = \psi_{200} + 2\psi_{211} + 3\psi_{210} + \sqrt{2}\psi_{21-1}$

The expectation value of L_z is given by

$$\langle L_z \rangle = \frac{\langle \psi | L_z | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\sum_l |C_l|^2 m_l \hbar}{\sum_l |C_l|^2} = \frac{1 \times 0 \times \hbar + 4 \times 2 \times \hbar + 9 \times 0 \times \hbar + 2 \times (-1) \times \hbar}{1 + 4 + 9 + 2} = \frac{4\hbar - 2\hbar}{16} = \frac{2\hbar}{16} = \frac{\hbar}{8}$$

Correct option is (d)

35. The energy eigenvalues of a particle in the potential $V(x) = \frac{1}{2}m\omega^2 x^2 - ax$ are

- (a) $E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{a^2}{2m\omega^2}$ (b) $E_n = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{a^2}{2m\omega^2}$
(c) $E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{a^2}{m\omega^2}$ (d) $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

Soln. The potential $V(x) = \frac{1}{2}m\omega^2 x^2 - ax$

The Schrodinger equation, $\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x)$

For S.E., $\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} \left(E - \frac{1}{2} m\omega^2 x^2 \right) \psi = 0 \quad \dots(1)$

The energy is given by $E = \left(n + \frac{1}{2} \right) \hbar\omega$

$$\Rightarrow \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 - ax \right] \psi(x) = E\psi(x)$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \left[E - \frac{1}{2} m\omega^2 x^2 + ax \right] \psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \left[E + \frac{a^2}{2m\omega^2} - \frac{1}{2} m\omega^2 x^2 + ax - \frac{a^2}{2m\omega^2} \right] \psi(x) = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \left[E + \frac{a^2}{2m\omega^2} - \frac{1}{2} m\omega^2 \left(x^2 + \frac{a^2}{m^2\omega^4} - \frac{2ax}{m\omega^2} \right) \right] \psi(x) = 0$$

Replace, $E + \frac{a^2}{2m\omega^2} = E'$, $x + \frac{a}{m\omega} = x'$ & $dx = dx'$

$$\Rightarrow \frac{d\psi}{dx'^2} + \frac{2m}{\hbar^2} \left[E' - \frac{1}{2} m\omega^2 x'^2 \right] \psi(x') = 0 \quad (2)$$

On comparing equation (2) by equation (1), the energy corresponding to equation (2) are given by

$$E' = \left(n + \frac{1}{2} \right) \hbar\omega \Rightarrow E + \frac{a^2}{2m\omega^2} = \left(n + \frac{1}{2} \right) \hbar\omega$$

$$\Rightarrow E = \left(n + \frac{1}{2} \right) \hbar\omega - \frac{a^2}{2m\omega^2}$$

Correct option is (a)

36. If a particle is represented by the normalized wave function

$$\psi(x) = \begin{cases} \frac{\sqrt{15}(a^2 - x^2)}{4a^{5/2}} & \text{for } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

the uncertainty Δp in its momentum is

(a) $\frac{2\hbar}{5a}$

(b) $\frac{5\hbar}{2a}$

(c) $\frac{\sqrt{10}\hbar}{a}$

(d) $\frac{\sqrt{5}\hbar}{\sqrt{2}a}$

Soln. The given normalized wavefunction is

$$\psi(x) = \begin{cases} \frac{\sqrt{15}(a^2 - x^2)}{4a^{5/2}} & \text{for } -a < x < +a \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \langle p \rangle &= \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{d}{dx} \right) \psi dx \\ &= \int_{-\infty}^{-a} 0 \left(-i\hbar \frac{d}{dx} \right) 0 dx + \int_{-a}^{+a} \frac{\sqrt{15}}{4a^{5/2}} (a^2 - x^2) \left(-i\hbar \frac{d}{dx} \right) \frac{\sqrt{15}}{4a^{5/2}} (a^2 - x^2) dx + \int_{+a}^{\infty} 0 \left(-i\hbar \frac{d}{dx} \right) 0 dx \\ &= -i\hbar \int_{-a}^{+a} \frac{15}{16a^5} (a^2 - x^2) (-2x) dx = 0 \quad [\text{As the function given in the integral is odd}] \end{aligned}$$

$$\begin{aligned} \langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi^* \left(-\hbar^2 \frac{d^2}{dx^2} \right) \psi dx \\ &= \int_{-\infty}^{-a} 0 \left(-\hbar^2 \frac{d^2}{dx^2} \right) 0 dx + \int_{-a}^{+a} \frac{15}{16a^5} (a^2 - x^2) \left(-\hbar^2 \frac{d^2}{dx^2} \right) (a^2 - x^2) dx + \int_{+a}^{\infty} 0 \left(-\hbar^2 \frac{d^2}{dx^2} \right) 0 dx \\ &= -\hbar^2 \int_{-a}^{+a} \frac{15}{16a^5} (a^2 - x^2) (-2) dx \\ &= \frac{15\hbar^2}{8a^5} \int_{-a}^{+a} (a^2 - x^2) dx = \frac{15\hbar^2}{8a^5} \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{+a} = \frac{15\hbar^2}{8a^5} \left[a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3} \right] = \frac{15}{8} \times \frac{4}{a^5} \times \frac{\hbar^2 a^3}{3} = \frac{5\hbar^2}{2a^2} \end{aligned}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{5\hbar^2}{2a^2} - 0} = \frac{\sqrt{5}\hbar}{\sqrt{2}a}$$

Correct option is (d)

37. Given the usual canonical commutation relations, the commutator $[A, B]$ of $A = i(xp_y - yp_x)$ and $B = (yp_z + zp_y)$ is:

(a) $\hbar(xp_z - p_x z)$ (b) $-\hbar(xp_z - p_x z)$ (c) $\hbar(xp_z + p_x z)$ (d) $-\hbar(xp_z + p_x z)$

Soln. $A = i(xp_y - yp_x)$ & $B = (yp_z + zp_y)$

$$[A, B] = [i(xp_y - yp_x), (yp_z + zp_y)]$$

By using the distributive properties,

$$[AB, C] = A[B, C] + [A, C]B \quad \& \quad [A, BC] = B[A, C] + [A, B]C$$

And associative properties $[A, B \pm C] = [A, B] \pm [A, C]$

We can simplify

$$\begin{aligned}
 [A, B] &= i[(x p_y - y p_x), y p_z] + i[(x p_y - y p_x), z p_y] \\
 &= i[x p_y, y p_z] - i[y p_x, y p_z] + i[x p_y, z p_y] - i[y p_x, z p_y]
 \end{aligned}$$

[* out of 4 commutation relations 2nd and 3rd will be zero as there will be no canonically conjugate pair term.

$$\begin{aligned}
 \Rightarrow [A, B] &= i[x p_y, y p_z] - i[y p_x, z p_y] \\
 &= i x [p_y, y p_z] + i [x, y p_z] p_y - i y [p_x, z p_y] - i [y, z p_y] p_x
 \end{aligned}$$

[2nd and 3rd will be also zero, reason will be same as above]

$$\begin{aligned}
 \Rightarrow [A, B] &= i x [p_y, y p_z] - i [y, z p_y] p_x \\
 &= i x y [p_y, p_z] + i x [p_y, y] p_z - i z [y, p_y] p_x - i [y, z] p_y p_x
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow [A, B] &= i x (-i \hbar) p_z - i z (i \hbar) p_x \\
 &= \hbar (x p_z - p_x z) \quad (\text{as } z p_y = p_y z) \\
 &= \hbar (x p_z - p_x z)
 \end{aligned}$$

Correct option is (c)

38. The entropy of a system, S , is related to the accessible phase space volume Γ by $S = k_B \ln \Gamma(E, N, V)$ where E , N and V are the energy, number of particles and volume respectively. From this one can conclude that Γ
- does not change during evolution to equilibrium
 - Oscillates during evolution to equilibrium
 - Is a maximum in equilibrium
 - Is a minimum in equilibrium

Soln. In the microcanonical ensemble, the phase volume is maximized to obtain the characteristic thermodynamics quantity which is entropy

Correct option is (c)

39. Let ΔW be the work done in a quasistatic reversible thermodynamics process. Which of the following statements about ΔW is correct?
- ΔW is a perfect differential if the process is isothermal
 - ΔW is a perfect differential if the process is adiabatic
 - ΔW is always a perfect differential
 - ΔW cannot be a perfect differential.

Soln. According to the first law of thermodynamics

$$dQ = dU + dW$$

where dQ is the amount of heat absorbed by a system, dU is the change in the internal energy of the system which depends only on the temperature, and dW is the amount of work done. If $dQ \neq 0$, then dW is not a perfect differential as it depends upon the path chosen. However for an adiabatic process $dQ = 0$, $\therefore dU = -dW$

Since dU is a perfect differential, therefore dW is also a perfect differential.

Correct option is (b)

40. Consider a system of three spins S_1, S_2 and S_3 each of which can take values $+1$ and -1 . The energy of the system is given by $E = -J[S_1S_2 + S_2S_3 + S_3S_1]$, where J is a positive constant. The minimum energy and the corresponding number of spin configurations are, respectively,
- (a) J and 1 (b) $-3J$ and 1 (c) $-3J$ and 2 (d) $-6J$ and 2

Soln. The energy of the system is given by

$E = -J[S_1S_2 + S_2S_3 + S_3S_1]$, where J is positive constant and each among S_1, S_2 and S_3 can take $+1$ and -1 values.

Suppose S_1, S_2, S_3 all have value $+1$, then the energy of the system is

$$E = -J(1 \times 1 + 1 \times 1 + 1 \times 1) = -3J$$

Suppose S_1, S_2, S_3 all have value -1 , then the energy of the system is

$$E = -J[(-1) \times (-1) + (-1) \times (-1) + (-1) \times (-1)] = -3J$$

If S_1, S_2, S_3 have different values then energy will be greater than $-3J$.

Correct option is (c)

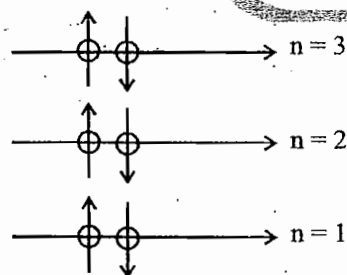
41. The minimum energy of a collection of 6 non-interacting electrons of spin $\frac{1}{2}$ placed in a one dimensional infinite square well potential of width L is

(a) $14\pi^2\hbar^2/mL^2$ (b) $9\pi^2\hbar^2/mL^2$ (c) $7\pi^2\hbar^2/mL^2$ (d) $3\pi^2\hbar^2/mL^2$

Soln. The energy of a particle of mass ' m ' in a one-dimension square well potential of width L is given by

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2}, \quad n = 1, 2, 3, 4, \dots$$

Only two electron with spin $\frac{1}{2}$ can fill in the same state.



(corresponding to minimum energy)

The minimum energy of the system is

$$= 2 \left(\frac{\hbar^2\pi^2}{2mL^2} + \frac{4\hbar^2\pi^2}{2mL^2} + \frac{9\hbar^2\pi^2}{2mL^2} \right) = \frac{14\pi^2\hbar^2}{mL^2}$$

Correct option is (a)

42. A live music broadcast consists of a radio-wave of frequency 7 MHz, amplitude-modulated by a microphone output consisting of signals with a maximum frequency of 10 KHz. The spectrum of modulated output will be zero outside the frequency band
- (a) 7.00 MHz to 7.01 MHz (b) 6.99 MHz to 7.01 MHz
(c) 6.99 MHz to 7.00 MHz (d) 6.995 MHz to 7.005 MHz

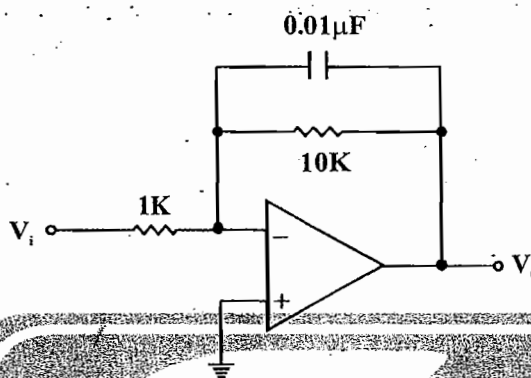
Soln. Carrier frequency $f_c = 7 \text{ MHz}$

Modulation frequency $f_m = 10 \text{ KHz} = 0.01 \text{ MHz}$

Modulated signal will have frequency components $(f_c + f_m) = 7.01 \text{ MHz}$ and $(f_c - f_m) = 6.99 \text{ MHz}$

Correct option is (b).

43. In the op-amp circuit shown in the figure, V_i is a sinusoidal input signal of frequency 10 Hz and V_o is the output signal.



The magnitude of the gain and the phase shift, respectively, are close to the values

- (a) $5\sqrt{2}$ and $\frac{\pi}{2}$ (b) $5\sqrt{2}$ and $\frac{\pi}{2}$ (c) 10 and zero (d) 10 and π

Soln. $\frac{V_o}{V_{in}} = \frac{-R_f}{R_i}$ [since inverting amplifier]

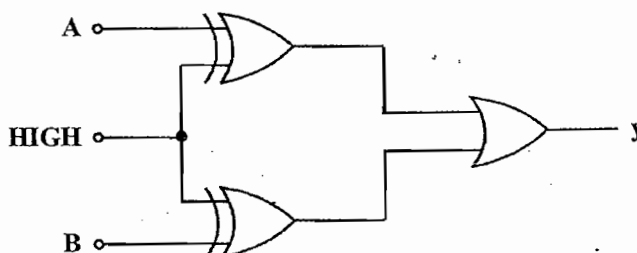
$$R_f = 10k \parallel 0.01\mu f = \frac{10k \times \frac{1}{\sqrt{\omega \times 0.01\mu}}}{10k + \frac{1}{\sqrt{\omega \times 0.01\mu}}} = \frac{10k}{1 + \sqrt{10 \times 2\pi \times 10 \times 10^3 \times 0.01 \times 10^{-6}}} = \frac{10k}{1 + \sqrt{6.28 \times 10^{-3}}}$$

$$\frac{V_o}{V_{in}} = \frac{-R_f}{R_i} = \frac{-10k}{\left(1 + \sqrt{6.28 \times 10^{-3}}\right)} \times \frac{1}{1k} = \frac{-10}{1 + \sqrt{6.28 \times 10^{-3}}} \Rightarrow \left| \frac{V_o}{V_{in}} \right| \approx 10$$

Phase difference is $180^\circ = \pi$

Correct options is (d)

44. The logic circuit shown in the figure below.



implements the Boolean expression

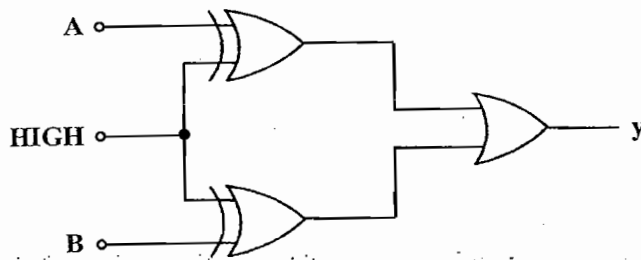
(a) $y = \overline{A.B}$

(b) $y = \overline{A.B}$

(c) $y = A.B$

(d) $y = A + B$

Soln.



High means 1.

Output of first EXOR gate is $\overline{A}$

Output of second EXOR gate is $\overline{B}$

Output of OR gate is $\overline{A} + \overline{B}$

As per demorgan's theorem $\overline{A.B} = \overline{A} + \overline{B}$

Correct options is (a).

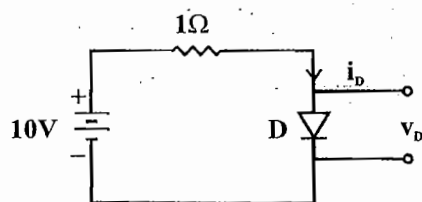
45. A diode D as shown in the circuit as an $i-v$ relation which can be proximated by

$$i_D = \begin{cases} v_D^2 + 2v_D, & \text{for } v_D > 0 \\ 0, & \text{for } v_D \leq 0 \end{cases}$$

The value of v_D in the circuit is:

- (a) $(-1 + \sqrt{11})V$ (b) $8V$ (c) $5V$ (d) $2V$

Soln.



Apply KVL

$$10 = i_D \times 1\Omega + v_D$$

[Given in question]

$$\Rightarrow 10 = 1\Omega \times [v_D^2 + 2v_D] + v_D$$

$$\Rightarrow v^2 + 3D + 10 = 0 \Rightarrow v_D = -5, 2 \text{ (neglect } -5)$$

Correct option is (d)

Part-C

46. The Taylor expansion of the function $\ln(\cosh x)$, where 'x' is real, about the point $x = 0$ starts with the following terms:

(a) $-\frac{1}{2}x^2 + \frac{1}{12}x^4 + \dots$

(b) $\frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots$

(c) $-\frac{1}{2}x^2 + \frac{1}{6}x^4 + \dots$

(d) $\frac{1}{2}x^2 + \frac{1}{6}x^4 + \dots$

Soln. Taylor series expansion of $f(x)$ about $x = x_0$ is:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots$$

Now, in question $x_0 = 0, f(x) = \ln(\cosh x)$

$$f'(x) = \frac{1}{\cosh x} \sinh x = \tanh x$$

$$f''(x) = \text{sech}^2 x;$$

$$f'''(x) = 2\text{sech } x (-\text{sech } x \cdot \tanh x) = -2\text{sech}^2 x \cdot \tanh x$$

$$f''''(x) = -2(2\text{sech } x)(-\text{sech } x \cdot \tanh x) \tanh x - 2\text{sech}^2 x \cdot \text{sech}^2 x = 4\text{sech}^2 x \cdot \tanh^2 x - 2\text{sech}^4 x$$

Required Taylor series expansion: $\ln(\cosh x) = 0 + 0 + \frac{1}{2}x^2 + 0 + \frac{1}{4!}(-2)x^4 + \dots$

$$\Rightarrow \ln(\cosh x) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots$$

Correct option is (b)

47. Given a 2×2 unitary matrix U satisfying $U^\dagger U = UU^\dagger = I$ with $\det U = e^{i\phi}$, one can construct a unitary matrix $V (V^\dagger V = VV^\dagger = I)$ with $\det V = 1$ from it by

(a) Multiplying U by $e^{-i\phi/2}$

(b) Multiplying any single element of U by $e^{-i\phi}$

(c) Multiplying any row or column of U by $e^{-i\phi/2}$

(d) Multiplying U by $e^{-i\phi}$

Soln. Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det U = ad - bc = e^{i\phi}$ (Given)

$$\text{Let, } V = e^{-i\phi/2} U \Rightarrow V = \begin{pmatrix} e^{-i\phi/2}a & e^{-i\phi/2}b \\ e^{-i\phi/2}c & e^{-i\phi/2}d \end{pmatrix}$$

$$\det V = e^{-i\phi} ad - e^{-i\phi} bc = e^{-i\phi} (ad - bc) = e^{-i\phi} e^{i\phi} = 1$$

We have to multiply U with $e^{-i\phi/2}$ to get V with determinant = 1

Correct option is (a)

48. The value of the integral $\int_C \frac{z^3 dz}{z^2 - 5z + 6}$, where C is a closed contour defined by the equation $2|z| - 5 = 0$, traversed in the anti-clockwise direction, is:
- (a) $-16\pi i$ (b) $16\pi i$ (c) $8\pi i$ (d) $2\pi i$

Soln. $f(z) = \frac{z^3}{z^2 - 5z + 6}$

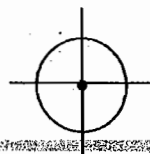
Poles: $z^2 - 5z + 6 = 0$

Given closed contour: $c: 2|z| - 5 = 0 \Rightarrow |z| = \frac{5}{2}$

$\Rightarrow (z-3)(z-2) = 0$

So, $z = 3 \Rightarrow \text{point } (3, 0)$

$z = 2 \Rightarrow \text{point } (2, 0)$



centre (0, 0), radius 5/2 units

Only $z = 2$ i.e. point (2, 0) is within 'c'

$\text{Res. } f(z) \Big|_{z=2} = (z-2) \frac{z^3}{(z-3)(z-2)} \Big|_{z=2} = -8$

Given integral: $I = 2\pi i \times (\text{sum of the residues}) = -16\pi i$

Correct option is (a)

49. A function $f(x)$ obeys the differential equation $\frac{d^2 f}{dx^2} - (3 - 2i)f = 0$ and satisfies the conditions $f(0) = 1$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. The value of $f(\pi)$ is:
- (a) $e^{2\pi}$ (b) $e^{-2\pi}$ (c) $-e^{-2\pi}$ (d) $-e^{2\pi}$

Soln. Given differential equation: $\frac{d^2 f}{dx^2} - (3 - 4i)f = 0$

(where 2 is wrongly printed in place of 4 in the question)

Let, $f = c.e^{mx}$ be trial solution.

$\Rightarrow \frac{df}{dx} = c.m e^{mx} \Rightarrow \frac{d^2 f}{dx^2} = c.m^2 e^{mx}$

Substituting in the given equation, we get

$m^2 - (3 - 4i) = 0 \quad (e^{mx} \neq 0)$

$\Rightarrow m = \sqrt{3 - 4i} = \sqrt{4 - 1 - 4i} = \sqrt{(2)^2 + (i)^2 - 2.2.i} = \sqrt{(2 - i)^2} = \pm(2 - i)$

Therefore, $f(x) = A e^{(2-i)x} + B e^{-(2-i)x}$

Since, $f(0) = 1 \Rightarrow A + B = 1$

Since, $f(x) \rightarrow 0$ when $x \rightarrow \infty \Rightarrow A = 0$, then $B = 1$

$$\therefore f(x) = e^{-(2-i)x} \Rightarrow f(\pi) = e^{-(2-i)\pi} = e^{-2\pi} e^{i\pi} \Rightarrow f(\pi) = -e^{-2\pi}$$

Correct option is (c).

50. A planet of mass 'm' moves in the gravitational field of the Sun (mass M). If the semi-major and semi-minor axes of the orbit are 'a' and 'b' respectively, the angular momentum of the planet is:

(a) $\sqrt{2GMm^2(a+b)}$

(b) $\sqrt{2GMm^2(a-b)}$

(c) $\sqrt{\frac{2GMm^2ab}{(a-b)}}$

(d) $\sqrt{\frac{2GMm^2ab}{(a+b)}}$

Soln. None of the option is correct.

Correct answer is $L = \sqrt{GMm^2b^2/a}$

51. The Hamiltonian of a simple pendulum consisting of a mass 'm' attached to a massless string of length l is $H = \frac{p_\theta^2}{2ml^2} + mg\ell(1 - \cos\theta)$. If L denotes the Lagrangian, the value of $\frac{dL}{dt}$ is:

(a) $-\frac{2g}{\ell} p_\theta \sin\theta$

(b) $-\frac{g}{\ell} p_\theta \sin 2\theta$

(c) $\frac{g}{\ell} p_\theta \cos\theta$

(d) $\ell p_\theta^2 \cos\theta$

Soln. $H = \frac{p_\theta^2}{2ml^2} + mg\ell(1 - \cos\theta) \Rightarrow L = \sum p_i q_i - H = p_\theta \dot{\theta} - H$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg \sin\theta$$

Therefore, $L = p_\theta \frac{p_\theta}{ml^2} - \left[\frac{p_\theta^2}{2ml^2} + mg\ell(1 - \cos\theta) \right] \Rightarrow L = \frac{p_\theta^2}{2ml^2} - mg\ell(1 - \cos\theta)$

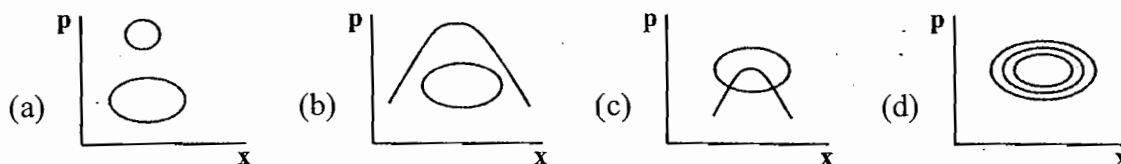
Since, $L = L(\theta, p_\theta)$ then $\frac{dL}{dt} = \frac{\partial L}{\partial p_\theta} \dot{p}_\theta + \frac{\partial L}{\partial \theta} \dot{\theta}$

Now, $\frac{\partial L}{\partial p_\theta} = \frac{p_\theta}{ml^2}$, $\dot{p}_\theta = -mg \sin\theta$, $\frac{\partial L}{\partial \theta} = -mg \sin\theta$, $\dot{\theta} = \frac{p_\theta}{ml^2}$

Then $\frac{dL}{dt} = \frac{p_\theta}{ml^2} (-mg \sin\theta) - mg \sin\theta \frac{p_\theta}{ml^2} = -\frac{2g}{\ell} p_\theta \sin\theta$

Correct option is (a)

52. Which of the following set of phase-space trajectories which one is not possible for a particle obeying Hamilton's equations of motion (for a time-independent Hamiltonian)?



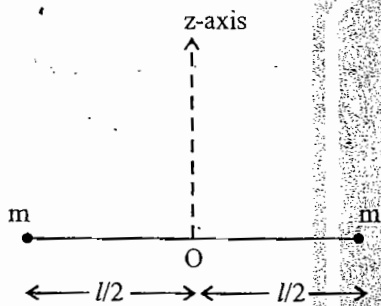
Soln. According to Louville's theorem, phase space trajectories cannot cross.

Correct option is (c)

53. Two bodies of equal mass 'm' are connected by a massless rigid rod of length 'l' lying in the xy-plane with the centre of the rod at the origin. If this system is rotating about the z-axis with a frequency ω , its angular momentum is

- (a) $m\ell^2\omega/4$ (b) $m\ell^2\omega/2$ (c) $m\ell^2\omega$ (d) $2m\ell^2\omega$

Soln.



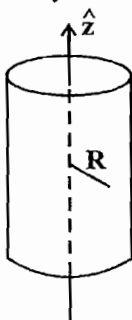
$L = \text{angular momentum} = I\omega$, where I is moment of inertia of the rod about an axis z with centre at O.

$$I = m\left(\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{m\ell^2}{4} + \frac{m\ell^2}{4} = \frac{m\ell^2}{2}$$

$$L = I\omega = \frac{m\ell^2\omega}{2}$$

Correct option is (b)

54. An infinite solenoid with its axis of symmetry along the z-direction carries a steady current I.



The vector potential $\vec{A}$ at a distance R from the axis.

- (a) Is constant inside and varies as R outside the solenoid.
 (b) Varies as R inside and is constant outside the solenoid.
 (c) Varies as $1/R$ inside and as R outside the solenoid.
 (d) Varies as R inside and as $1/R$ outside the solenoid.

Soln. Infinite solenoid carrying a current I . The vector potential $\vec{A}$ at a distance R from the axis, then

$$\text{We know that, } \oint \vec{A} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \int_S \vec{B} \cdot d\vec{S}$$

Let radius of solenoid is r .

$$\text{then inside, } \int_S \vec{B} \cdot d\vec{S} = B \cdot \pi R^2 \text{ \& } \oint \vec{A} \cdot d\vec{\ell} = A \cdot 2\pi R$$

$$\text{Hence, } A \cdot 2\pi R = B \pi R^2$$

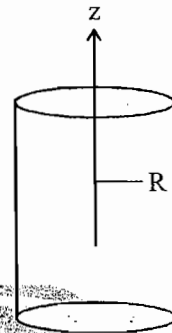
$$A = \frac{BR}{2} \text{ but } B = \mu_0 n I$$

$$\text{Hence, } A = \frac{\mu_0 n I R}{2} \quad A_{\text{in}} \propto R$$

$$\text{Outside, } A \cdot 2\pi R = B \cdot \pi r^2$$

$$A_{\text{out}} = \frac{\mu_0 n I r^2}{2R}$$

$$A_{\text{out}} \propto \frac{1}{R}$$



Correct option is (d).

55. Consider an infinite conducting sheet in the xy -plane with a time dependent current density $Kt\hat{i}$, where K is a constant. The vector potential at (x, y, z) is given by

$$\vec{A} = \frac{\mu_0 K}{4c} (ct - z)^2 \hat{i}$$

The magnetic field $\vec{B}$ is

$$(a) \frac{\mu_0 K t}{2} \hat{j}$$

$$(b) \frac{\mu_0 K z}{2c} \hat{j}$$

$$(c) \frac{\mu_0 K}{2c} (ct - z) \hat{i}$$

$$(d) \frac{\mu_0 K}{2c} (ct - z) \hat{j}$$

Soln. Vector potential

$$\vec{A} = \frac{\mu_0 k}{4c} (ct - z)^2 \hat{i}$$

$$\text{We know that, magnetic field } \vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & 0 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j} \left[0 - \frac{\partial}{\partial z} A_x \right] + \hat{k} \left[0 - \frac{\partial}{\partial y} A_x \right]$$

$$= +\hat{j} \frac{\partial}{\partial z} \left[\frac{\mu_0 k}{4c} (ct - z)^2 \right] = \frac{\mu_0 k}{2c} (ct - z)(-1) \hat{j} = -\frac{\mu_0 k}{2c} (ct - z) \hat{j}$$

Correct option is (d).

56. When a charged particle emits electromagnetic radiation, the electric field $\vec{E}$ and the Poynting vector $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ at a large distance 'r' from the emitter vary as $\frac{1}{r^n}$ and $\frac{1}{r^m}$ respectively. Which of the following choices for 'n' and 'm' are correct?
 (a) $n=1$ and $m=1$ (b) $n=2$ and $m=2$ (c) $n=1$ and $m=2$ (d) $n=2$ and $m=4$

Soln. For electromagnetic radiation

$$E \propto \frac{1}{r} \text{ and } B \propto \frac{1}{r}. \text{ Hence, poynting vector } S = \frac{\vec{E} \times \vec{B}}{\mu_0} \propto \frac{1}{r^2}$$

Hence, $n=1$ $m=2$

Correct option is (c).

57. The energies in the ground state and first excited state of a particle of mass $m = \frac{1}{2}$ in a potential $V(x)$ are -4 and -1 , respectively, (in units in which $\hbar = 1$). If the corresponding wavefunctions are related by $\psi_1(x) = \psi_0(x) \sinh x$, then the ground state eigenfunction is
 (a) $\psi_0(x) = \sqrt{\sec hx}$ (b) $\psi_0(x) = \sec hx$
 (c) $\psi_0(x) = \sec h^2 x$ (d) $\psi_0(x) = \sec h^3 x$

Soln. The time independent schrodinger equation $\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x)$

On taking $m = \frac{1}{2}, \hbar = 1$

$$\left[-\frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x)$$

For ground state, $\left[-\frac{d^2}{dx^2} + V(x) \right] \psi_0(x) = -4\psi_0(x)$

$$V(x)\psi_0(x) = \frac{d^2\psi_0}{dx^2} - 4\psi_0(x) \quad \dots (1)$$

For first excited state, $\left[-\frac{d^2}{dx^2} + V(x) \right] \psi_1(x) = -\psi_1(x)$

Since, $\psi_1(x) = \psi_0(x) \sinh x$

$$\Rightarrow \left[-\frac{d^2}{dx^2} + V(x) \right] \psi_0(x) \sinh x = -\psi_0(x) \sinh x$$

$$\Rightarrow -\frac{d^2}{dx^2} (\psi_0(x) \sinh x) + V(x) \psi_0(x) \sinh x = -\psi_0(x) \sinh x$$

$$\Rightarrow -\frac{d}{dx} \left[\psi_0(x) \cosh x + \sinh x \frac{d\psi_0}{dx} \right] + V(x) \psi_0 \sinh x = -\psi_0(x) \sinh x$$

$$-\psi_0(x) \sinh x - \cosh x \frac{d\psi_0}{dx} - \cosh x \frac{d\psi_0}{dx} - \sinh x \frac{d^2\psi_0}{dx^2} + V(x) \psi_0(x) \sinh x = -\psi_0(x) \sinh x$$

$$\Rightarrow -2 \cosh x \frac{d\psi_0}{dx} - \sinh x \frac{d^2\psi_0}{dx^2} + V(x) \psi_0(x) \sinh x = 0$$

$$\Rightarrow \frac{d^2\psi_0}{dx^2} + \frac{2 \cosh x}{\sinh x} \frac{d\psi_0}{dx} = V(x) \psi_0(x) \quad \dots (2)$$

Substituting the value of $V(x) \psi_0(x)$ from equation (1) into equation (2)

$$\Rightarrow \frac{d^2\psi_0}{dx^2} + \frac{2 \cosh x}{\sinh x} \frac{d\psi_0}{dx} = \frac{d^2\psi_0}{dx^2} - 4\psi_0(x)$$

$$\Rightarrow \frac{d\psi_0}{dx} = -2 \frac{\sinh x}{\cosh x} \psi_0(x); \quad \frac{d\psi_0}{\psi_0(x)} = -\frac{2 \sinh x}{\cosh x} dx$$

$$\ln \psi_0(x) = -2 \ln \cosh x = \ln \frac{1}{\cosh^2 x} \Rightarrow \ln \psi_0(x) = \ln \operatorname{sech}^2 x$$

$$\boxed{\psi_0(x) = \operatorname{sech}^2 x}$$

Correct option is (c).

58. The perturbation

$$H' = \begin{cases} b(a-x) & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

acts on a particle of mass 'm' confined in an infinite square well potential

$$V(x) = \begin{cases} 0 & -a < x < a \\ \infty & \text{otherwise} \end{cases}$$

The first order correction to the ground state energy of the particle is

(a) $\frac{ba}{2}$

(b) $\frac{ba}{\sqrt{2}}$

(c) $2ba$

(d) ba

Soln. The ground state wavefunction for the given potential $\psi(x) = \sqrt{\frac{2}{2a}} \cos\left(\frac{\pi x}{2a}\right) = \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right)$

The perturbation, $H' = \begin{cases} b(a-x) & -a < x < a \\ 0 & \text{otherwise} \end{cases}$

The first order correction in energy,

$$E' = \int \psi^* H' \psi dx$$

$$= \int_{-a}^{+a} \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right) b(a-x) \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right) dx = \frac{b}{a} \int_{-a}^{+a} \cos^2\left(\frac{\pi x}{2a}\right) (a-x) dx$$

$$= \frac{b}{a} \int_{-a}^{+a} \frac{1 + \cos\left(\frac{\pi x}{a}\right)}{2} (a-x) dx = \frac{b}{2a} \left[\int_{-a}^{+a} (a-x) dx + \int_{-a}^{+a} a \cos\left(\frac{\pi x}{a}\right) dx - \int_{-a}^{+a} x \cos\left(\frac{\pi x}{a}\right) dx \right]$$

as integral is odd

$$\frac{b}{2a} \left[\left\{ ax - \frac{x^2}{2} \right\} + \frac{a \sin\left(\frac{\pi x}{a}\right)}{\frac{\pi}{a}} \right]_{-a}^{+a} = \frac{b}{2a} [2a^2 + 0] = ab$$

Correct option is (d)

59. Let $|0\rangle$ and $|1\rangle$ denote the normalized eigenstates corresponding to the ground and the first excited states of a one-dimensional harmonic oscillator. The uncertainty Δx in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is:

(a) $\Delta x = \sqrt{\hbar/2m\omega}$ (b) $\Delta x = \sqrt{\hbar/m\omega}$ (c) $\Delta x = \sqrt{2\hbar/m\omega}$ (d) $\Delta x = \sqrt{\hbar/4m\omega}$

Soln. $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\langle\psi| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$

$$\langle\hat{x}\rangle = \langle\psi|\hat{x}|\psi\rangle = \frac{1}{2} [\langle 0|\hat{x}|0\rangle + \langle 1|\hat{x}|1\rangle + \langle 0|\hat{x}|1\rangle + \langle 1|\hat{x}|0\rangle]$$

Now, $\langle 0|\hat{x}|0\rangle = \langle 1|\hat{x}|1\rangle = 0$ [as the states of LHO have definite parity. So, the expectation value of odd operator will become zero]

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \quad [\text{where } \hat{a} \text{ \& \; } \hat{a}^\dagger \text{ are lowering and raising operators respectively}]$$

$$\begin{aligned} \left\langle 0 \left| \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \right| 0 \right\rangle &= \sqrt{\frac{\hbar}{2m\omega}} (\langle 0|\hat{a}|0\rangle + \langle 0|\hat{a}^\dagger|0\rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}} [\langle 0|0\rangle + \langle 0|1\rangle] = \sqrt{\frac{\hbar}{2m\omega}} [0 + 0] = 0 \end{aligned}$$

Similarly, $\langle 1|\hat{x}|1\rangle = 0$

$$\begin{aligned} \langle 0|\hat{x}|1\rangle &= \sqrt{\frac{\hbar}{2m\omega}} [\langle 0|\hat{a}|1\rangle + \langle 0|\hat{a}^\dagger|1\rangle] = \sqrt{\frac{\hbar}{2m\omega}} [\langle 0|1\rangle + \langle 0|\sqrt{2}|2\rangle] \\ &= \sqrt{\frac{\hbar}{2m\omega}} [1\langle 0|0\rangle + \sqrt{2}\langle 0|2\rangle] \quad [\text{using } \langle n|m\rangle = \delta_{mn}] \\ &= \sqrt{\frac{\hbar}{2m\omega}} \end{aligned}$$

$$\langle 1|\hat{x}|0\rangle = (\langle 0|\hat{x}|1\rangle)^\dagger = \sqrt{\frac{\hbar}{2m\omega}} \Rightarrow \langle \hat{x} \rangle = \left(\frac{1}{2} + \frac{1}{2}\right) \sqrt{\frac{\hbar}{2m\omega}} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\text{Therefore, } \langle \hat{x}^2 \rangle = \langle \psi | \hat{x}^2 | \psi \rangle = \frac{1}{2} [\langle 0 | \hat{x}^2 | 0 \rangle + \langle 1 | \hat{x}^2 | 1 \rangle + \langle 1 | \hat{x}^2 | 0 \rangle + \langle 0 | \hat{x}^2 | 1 \rangle]$$

$$\begin{aligned} \left\langle 0 \left| \frac{\hbar}{2m\omega} (\hat{a} + \hat{a}^\dagger)^2 \right| 0 \right\rangle &= \frac{\hbar}{2m\omega} \left[\langle 0 | (\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) | 0 \rangle \right] \\ &= \frac{\hbar}{2m\omega} [\langle 0 | \hat{a}^2 | 0 \rangle + \langle 0 | \hat{a}^{\dagger 2} | 0 \rangle + \langle 0 | (2\hat{a}^\dagger\hat{a} + 1) | 0 \rangle] \\ &= \frac{\hbar}{2m\omega} [\langle 0 | 0 \rangle + \langle 0 | \sqrt{1}\sqrt{2} | 2 \rangle + 1 \langle 0 | 0 \rangle] \\ &= \frac{\hbar}{2m\omega} [0 \langle 0 | 0 \rangle + \sqrt{2} \langle 0 | 2 \rangle + 1 \langle 0 | 0 \rangle] = \frac{\hbar}{2m\omega} \quad [\text{Using } \langle n | m \rangle = \delta_{mn}] \end{aligned}$$

$$\left\langle 1 \left| \frac{\hbar}{2m\omega} (\hat{a} + \hat{a}^\dagger)^2 \right| 1 \right\rangle = \frac{\hbar}{2m\omega} \left[\langle 1 | \hat{a}^2 | 1 \rangle + \langle 1 | \hat{a}^{\dagger 2} | 1 \rangle + \langle 1 | (2\hat{a}^\dagger\hat{a} + 1) | 1 \rangle \right] = \frac{3}{2} \frac{\hbar}{m\omega}$$

$$\langle 0 | \hat{x}^2 | 1 \rangle = \langle 1 | \hat{x}^2 | 0 \rangle = 0 \Rightarrow \langle \hat{x}^2 \rangle = \frac{1}{2} \left[\frac{\hbar}{2m\omega} + \frac{3}{2} \frac{\hbar}{m\omega} \right] = \frac{\hbar}{m\omega}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle \hat{x} \rangle^2} = \sqrt{\frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}} = \sqrt{\frac{\hbar}{2m\omega}}$$

Correct option is (a)

60. What would be the ground state energy of the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha \delta(x)$$

if vibrational principle is used to estimate it with the trial wavefunction $\psi(x) = A e^{-bx^2}$ with b as the variational parameter?

$$[\text{Hint: } \int_{-\infty}^{\infty} x^{2n} e^{-2bx^2} dx = (2b)^{-n-\frac{1}{2}} \Gamma\left(n + \frac{1}{2}\right)]$$

- (a) $-\alpha^2/2\hbar^2$ (b) $-2\alpha^2/\pi\hbar^2$ (c) $-\alpha^2/\pi\hbar^2$ (d) $\alpha^2/\pi\hbar^2$

Soln. The Hamiltonian, $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha \delta(x)$

The trial wave function $\psi(x) = A e^{-bx^2}$ with b variational parameter.

$$\begin{aligned}
 \langle H \rangle &= \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int_{-\infty}^{+\infty} A^* e^{-bx^2} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha \delta(x) \right) A e^{-bx^2} dx}{\int_{-\infty}^{+\infty} A^* e^{-bx^2} A e^{-bx^2} dx} \\
 &= \frac{|A|^2 \int_{-\infty}^{+\infty} e^{-bx^2} \left(\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) e^{-bx^2} dx - \alpha \int_{-\infty}^{+\infty} e^{-2bx^2} \delta(x) dx \right)}{|A|^2 \int_{-\infty}^{+\infty} e^{-2bx^2} dx} \\
 &= \frac{-\frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} e^{-bx^2} (4b^2 x^2 - 2b) e^{-bx^2} dx - \alpha}{\int_{-\infty}^{+\infty} e^{-2bx^2} dx} \quad \left[\text{by using } \int_{-\infty}^{+\infty} f(x) \delta(x-a) dx = f(a) \right] \\
 \langle \hat{H} \rangle &= \frac{-\frac{\hbar^2}{2m} 4b^2 \int_{-\infty}^{+\infty} e^{-2bx^2} x^2 dx + \frac{\hbar^2}{2m} 2b \int_{-\infty}^{+\infty} e^{-2bx^2} dx - \alpha}{\int_{-\infty}^{+\infty} e^{-2bx^2} dx} = \frac{-\frac{4b^2 \hbar^2}{2m} \int_0^{\infty} e^{-2bx^2} x^2 dx + \frac{2b \hbar^2}{2m} \int_0^{\infty} e^{-2bx^2} dx - \frac{\alpha}{2}}{\int_0^{\infty} e^{-2bx^2} dx} \\
 &= \frac{-\frac{4b^2 \hbar^2}{2m} \times \frac{1}{2} \frac{\sqrt{\pi/2}}{(2b)^{3/2}} + \frac{2b \hbar^2}{2m} \times \frac{\sqrt{\pi}}{2(2b)^{1/2}} - \frac{\alpha}{2}}{\frac{\sqrt{\pi}}{2(2b)^{1/2}}} = -\frac{\hbar^2 b}{2m} + \frac{2b \hbar^2}{2m} - \frac{\alpha}{\sqrt{\pi} (2b)^{1/2}} = \frac{b \hbar^2}{2m} - \frac{\alpha (2b)^{1/2}}{\sqrt{\pi}} \\
 \frac{\partial \langle H \rangle}{\partial b} &= 0 \Rightarrow \frac{\hbar^2}{2m} - \frac{\alpha \times 2^{1/2}}{\sqrt{\pi} 2b^{1/2}} = 0 \Rightarrow \frac{\alpha}{\sqrt{2\pi b}} = \frac{\hbar^2}{2m} \\
 &\Rightarrow \frac{\alpha^2}{2\pi b} = \frac{\hbar^4}{4m^2} \Rightarrow b = \frac{2\alpha^2 m^2}{\pi \hbar^4}
 \end{aligned}$$

The ground state energy,

$$\begin{aligned}
 E &= \frac{\hbar^2}{2m} \left(\frac{2\alpha^2 m^2}{\pi \hbar^4} \right) - \frac{\alpha \times 2^{1/2}}{\sqrt{\pi}} \left(\frac{2\alpha^2 m^2}{\pi \hbar^4} \right)^{1/2} \\
 &= \frac{\alpha^2 m}{\pi \hbar^2} - \frac{2\alpha^2 m}{\pi \hbar^2} = -\frac{\alpha^2 m}{\pi \hbar^2}
 \end{aligned}$$

Correct option is (c)

61. The free energy difference between the superconducting and the normal states of a material is given by $\Delta F = F_S - F_N = \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4$, where ψ is an order parameter and α and β are constants such that $\alpha > 0$ in the normal and $\alpha < 0$ in the superconducting state, while $\beta > 0$ always. The minimum value of ΔF in the superconducting state is

- (a) $-\alpha^2 / \beta$ (b) $-\alpha^2 / 2\beta$ (c) $-3\alpha^2 / 2\beta$ (d) $-5\alpha^2 / 2\beta$

Soln. $\Delta F = \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4$

Let $|\psi| = x$; $\Delta F = \alpha x^2 + \frac{\beta}{2} x^4$

$\frac{\partial}{\partial x}(\Delta F) = 2\alpha x + 2\beta x^3 = 0$; $2x(\alpha + \beta x^2) = 0$

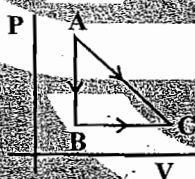
$\therefore x = 0$ (trivial solution)

$\alpha + \beta x^2 = 0 \Rightarrow x^2 = -\frac{\alpha}{\beta}$

$\therefore (\Delta F)_{\min} = \left| \alpha x^2 + \frac{\beta}{2} x^4 \right|_{x^2 = -\frac{\alpha}{\beta}} = \alpha \left(-\frac{\alpha}{\beta} \right) + \frac{\beta}{2} \frac{\alpha^2}{\beta^2} = -\frac{\alpha^2}{\beta} + \frac{1}{2} \frac{\alpha^2}{\beta} = -\frac{\alpha^2}{2\beta}$

Correct option is (b)

62. A given quantity of gas is taken from the state A $\rightarrow$ C reversibly, by two paths, A $\rightarrow$ C directly and A $\rightarrow$ B $\rightarrow$ C as shown in the figure below.



During the A $\rightarrow$ C the work done by the gas is 100 J and the heat absorbed is 150 J. If during the process A $\rightarrow$ B $\rightarrow$ C the work done by the gas is 30 J the heat absorbed is

- (a) 20 J (b) 80 J (c) 220 J (d) 280 J

Soln. $dQ = dU + dW_1$, $dW_1 = 100 \text{ J}$

For path AC: $150 \text{ J} = dU + 100 \text{ J}$
 $dU = 150 \text{ J} - 100 \text{ J} = 50 \text{ J}$

For path ABC: $dQ = dU + dW_2 = 50 \text{ J} + 30 \text{ J} = 80 \text{ J}$

Correct option is (b)

63. Consider a one-dimensional Ising model with N spins, at very low temperatures when almost all the spins are aligned parallel to each other. There will be a few spin flips with each flip costing an energy 2J. In a configuration with r spin flips, the energy of the system is $E = -NJ + 2rJ$ and the number of configuration is ${}^N C_r$; r varies from 0 to N. The partition function is

- (a) $\left(\frac{J}{k_B T} \right)^N$ (b) $e^{-NJ/k_B T}$ (c) $\left(\sinh \frac{J}{k_B T} \right)^N$ (d) $\left(\cosh \frac{J}{k_B T} \right)^N$

Soln. Let us consider only three energy levels, $E_r = -2J + 2rJ$
 $E_0 = -2J, E_1 = 0, E_2 = 2J$

$$Q_2 = \left({}^2C_0 e^{-\beta E_0} + {}^2C_1 e^{-\beta E_1} + {}^2C_2 e^{-\beta E_2} \right) / \sum_{r=0}^2 {}^2C_r$$

$$= \frac{(e^{\beta 2J} + 2e^0 + e^{-\beta 2J})}{4} = \frac{(e^{\beta J} + e^{-\beta J})^2}{4} = \left(\frac{e^{\beta J} + e^{-\beta J}}{2} \right)^2$$

$$Q_2 = (\cosh \beta J)^2 \quad \therefore Q_N = (\cosh \beta J)^N = \left(\cosh \frac{J}{kT} \right)^N$$

Correct option is (d)

64. A magnetic field sensor based on the Hall effect is to be fabricated by implanting. As into a Si film of thickness $1 \mu\text{m}$. The specifications require a magnetic field sensitivity of 500 mV/Tesla at an excitation current of 1 mA . The implantation dose is to be adjusted such that the average carrier density, after activation, is
- (a) $1.25 \times 10^{26} \text{ m}^{-3}$ (b) $1.25 \times 10^{22} \text{ m}^{-3}$ (c) $4.1 \times 10^{21} \text{ m}^{-3}$ (d) $4.1 \times 10^{20} \text{ m}^{-3}$

Soln. Hall voltage to magnetic field ratio $\frac{V_H}{B} = 500 \times 10^{-3} \text{ V/T}$

Current, $I = 1 \text{ mA} = 10^{-3} \text{ Amp}$

Thickness of sample, $t = 1 \times 10^{-6} \text{ m}$

We know that Hall voltage, $V_H = \frac{IB}{ne}$

$$\Rightarrow \text{concentration of dopants, } n = \frac{IB}{te V_H} = \frac{10^{-3}}{10^{-6} \times 1.6 \times 10^{-19} \times 500 \times 10^{-3}}$$

$$= \frac{100 \times 10^{-21}}{5 \times 1.6} = \frac{100}{8} \times 10^{21} = 1.25 \times 10^{22} / \text{m}^3$$

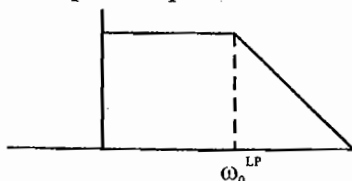
Correct option is (b).

65. Band-pass and band-reject filters can be implemented by combining a low pass and a high pass filter in series and in parallel, respectively. If the cut-off frequencies of the low pass and high pass filters are ω_0^{LP} and ω_0^{HP} , respectively, the condition required to implement the band-pass and band-reject filters are respectively.
- (a) $\omega_0^{\text{HP}} < \omega_0^{\text{LP}}$ and $\omega_0^{\text{HP}} < \omega_0^{\text{LP}}$ (b) $\omega_0^{\text{HP}} < \omega_0^{\text{LP}}$ and $\omega_0^{\text{HP}} > \omega_0^{\text{LP}}$
- (c) $\omega_0^{\text{HP}} > \omega_0^{\text{LP}}$ and $\omega_0^{\text{HP}} < \omega_0^{\text{LP}}$ (d) $\omega_0^{\text{HP}} > \omega_0^{\text{LP}}$ and $\omega_0^{\text{HP}} > \omega_0^{\text{LP}}$

Soln. Low pass cut off frequency ω_0^{LP}

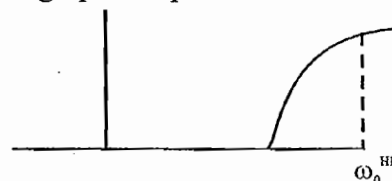
High pass out off frequency ω_0^{HP}

Low pass response.

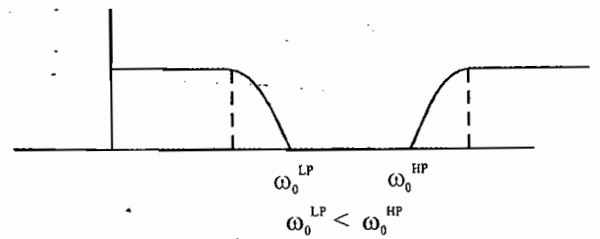
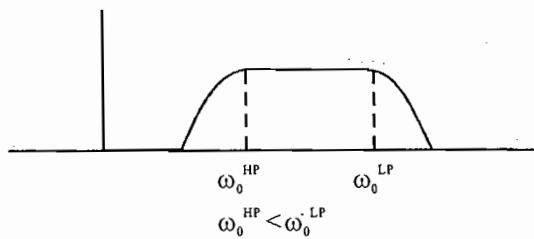


For band pass filter responses is:

High pass responses

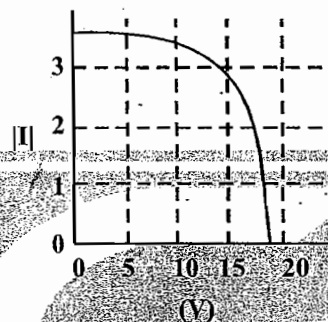


For band reject filter response is



Correct option is (b).

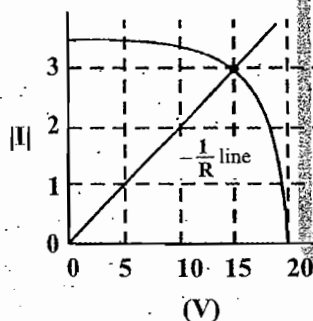
66. The output characteristics of a solar panel at a certain level of irradiance is shown in the figure below.



If the solar cell is to power a load of $5\ \Omega$, the power drawn by the load is

- (a) 97 W (b) 73 W (c) 50 W (d) 45 W

Soln.



For $5\ \Omega$ resistance

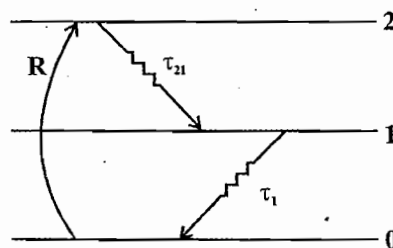
$$R = \frac{V}{I}$$

power drawn is

$$P = 3 \times 15 = 45 \text{ watt}$$

Hence, correct option is (d).

67. Consider the energy level diagram shown below, which corresponds to the molecular nitrogen laser.



If the pump rate R is $10^{20} \text{ atoms cm}^{-3} \text{ s}^{-1}$ and the decay routes are as shown with $\tau_{21} = 20 \text{ ns}$ and

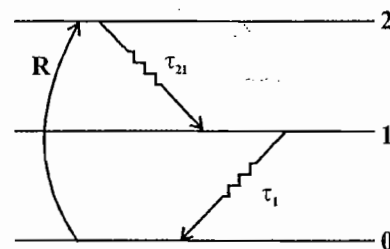
$\tau_1 = 1 \mu\text{s}$, the equilibrium populations of states 2 and 1 are, respectively,

- (a) 10^{14} cm^{-3} and $2 \times 10^{12} \text{ cm}^{-3}$ (b) $2 \times 10^{12} \text{ cm}^{-3}$ and 10^{14} cm^{-3}
(c) $2 \times 10^{12} \text{ cm}^{-3}$ and $2 \times 10^6 \text{ cm}^{-3}$ (d) zero and 10^{20} cm^{-3} .

$$R = 10^{20} \text{ atoms/cm}^3 - \text{sec}$$

$$\tau_{21} = 20 \times 10^{-9} \text{ sec}$$

$$\tau_1 = 1 \times 10^{-6} \text{ sec}$$



Rate equations for level (1) and (2) are given by

$$\frac{dN_2}{dt} = R - \frac{N_2}{\tau_{21}} \quad \dots (1)$$

$$\frac{dN_1}{dt} = \frac{N_2}{\tau_{21}} - \frac{N_1}{\tau_1} \quad \dots (2)$$

Under equilibrium, $\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0$

$$N_2 = R\tau_{21} = 10^{20} \times 20 \times 10^{-9} = 2 \times 10^{12} \text{ cm}^{-3}$$

$$N_1 = \frac{\tau_1 N_2}{\tau_{21}} = \frac{10^{-6} \times 10^{12} \times 2}{20 \times 10^{-9}} = 10^{14} \text{ cm}^{-3}$$

Correct option is (b).

68. Consider a hydrogen atom undergoing a $2P \rightarrow 1S$ transition. The lifetime τ_{sp} of the $2P$ state for spontaneous emission is 1.6 ns and the energy difference between the levels is 10.2 eV . Assuming that the refractive index of the medium $n_0 = 1$, the ratio of the Einstein coefficients for stimulated emission $B_{21}(\omega)/A_{21}(\omega)$ is given by

(a) $0.683 \times 10^{12} \text{ m}^3 \text{ J}^{-1} \text{ s}^{-1}$

(b) $0.146 \times 10^{-12} \text{ J s m}^{-3}$

(c) $6.83 \times 10^{12} \text{ m}^3 \text{ J}^{-1} \text{ s}^{-1}$

(d) $1.463 \times 10^{-12} \text{ J s m}^{-3}$

Soln.

$$n_0 = 1$$

$$\Delta E = 10.2 \text{ eV}$$

$$\begin{aligned} \frac{B_{21}}{A_{21}} &= \frac{\pi^2 c^3}{\hbar \omega^3 n_0^3} = \frac{\hbar^2 \pi^2 c^3}{(\Delta E)^3 n_0^3} = \frac{(3.14)^2 \times (3)^3 \times 10^{24} \times (6.6)^2 \times 10^{-68}}{(10.2)^3 \times (1.6)^3 \times 10^{-57} \times (1)^3 \times (3.14)^2 \times 4} \\ &= 0.0676 \times 10^{13} = 0.676 \times 10^{12} \end{aligned}$$

Correct option is (a)

69. Consider a He-Ne laser cavity consisting of two mirrors of reflectivities $R_1 = 1$ and $R_2 = 0.98$. The mirrors are separated by a distance $d = 20 \text{ cm}$ and the medium in between has a refractive index $n_0 = 1$ and absorption coefficient $\alpha = 0$. The values of the separation between the modes δv and the width Δv_p of each mode of the laser cavity are:

(a) $\delta v = 75 \text{ kHz}$, $\Delta v_p = 24 \text{ kHz}$

(b) $\delta v = 100 \text{ kHz}$, $\Delta v_p = 100 \text{ kHz}$

(c) $\delta v = 750 \text{ MHz}$, $\Delta v_p = 2.4 \text{ MHz}$

(d) $\delta v = 2.4 \text{ MHz}$, $\Delta v_p = 750 \text{ MHz}$

Soln. Laser mode separation, $\delta\nu = \frac{c}{2dn_0}$

but $c = 3 \times 10^8$ m/sec, $d = 20$ cm $= 20 \times 10^{-2}$ m

$$\boxed{n_0 = 1} \Rightarrow \delta\nu = \frac{3 \times 10^8}{2 \times 20 \times 10^{-2} \times 1} = \frac{30}{4} \times 10^8 \text{ Hz} = 7.5 \times 10^8 \text{ Hz} = \boxed{750 \text{ MHz}}$$

Width of each mode, $\Delta\nu_p = \frac{1}{2\pi t_c}$

where t_c = cavity lifetime $= \frac{2n_0 d}{c \ln\left(\frac{1}{1-x}\right)}$ and $x = 1 - R_1 R_2 e^{-2\alpha_c d}$

$R_1 = 1, R_2 = 0.98$

$$\boxed{\alpha_c = 0}, \quad x = 1 - 1 \times 0.98 = \boxed{0.02}$$

$$t_c = \frac{2 \times 1 \times 20 \times 10^{-2}}{3 \times 10^8 \ln\left(\frac{1}{0.98}\right)} = \frac{2 \times 20 \times 10^{-2}}{3 \times 10^8 \times 0.020} = \frac{2}{3} \times 10^{-7} \text{ sec}$$

$$\Delta\nu_p = \frac{3}{2 \times 3.14 \times 2 \times 10^{-7}} = 2.38 \times 10^6 \text{ Hz} \approx \boxed{2.4 \text{ MHz}}$$

Correct option is (c)

70. Non-interacting bosons undergo Bose-Einstein Condensation (BEC) when trapped in a three-dimensional isotropic simple harmonic potential. For BEC to occur, the chemical potential must be equal to
- (a) $\hbar\omega/2$ (b) $\hbar\omega$ (c) $3\hbar\omega/2$ (d) 0

Soln. For 3-D isotropic harmonic potential

$E_n = \left(n + \frac{3}{2}\right)\hbar\omega$; $E_0 = \frac{3}{2}\hbar\omega$ is the energy of the ground state. When BEC occurs, all the non-interacting bosons go to the ground state. The chemical potential = ground state of the bosonic system = $\frac{3}{2}\hbar\omega$

Correct option is (c)

71. In a band structure calculation, the dispersion relation for electrons is found to be

$$\epsilon_k = \beta(\cos k_x a + \cos k_y a + \cos k_z a)$$

where β is a constant and a is the lattice constant. The effective mass at the boundary of the first Brillouin zone is

- (a) $\frac{2\hbar^2}{5\beta a^2}$ (b) $\frac{4\hbar^2}{5\beta a^2}$ (c) $\frac{\hbar^2}{2\beta a^2}$ (d) $\frac{\hbar^2}{3\beta a^2}$

Soln. $\epsilon_k = \beta (\cos k_x a + \cos k_y a + \cos k_z a)$

This is tight-binding energy expression for simple cubic solid with lattice parameter a . The expression for effective mass (m^*) is given by

$$m^* = \frac{\hbar^2}{\left(\frac{d^2 \epsilon_k}{dk^2} \right)}$$

Brillouin zone boundary is at $k_x = \pm \frac{\pi}{a}, k_y = \pm \frac{\pi}{a}, k_z = \pm \frac{\pi}{a}$

Hence, $\left(\frac{d^2 \epsilon_k}{dk^2} \right)_{\frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a}} = 3\beta a^2$

Hence, $m^* = \frac{\hbar^2}{3\beta a^2}$

Correct option is (d)

72. The radius of the Fermi sphere of free electrons in a monovalent metal with an fcc structure, in which the volume of the unit cell is a^3 , is

(a) $\left(\frac{12\pi^2}{a^3} \right)^{1/3}$ (b) $\left(\frac{3\pi^2}{a^3} \right)^{1/3}$ (c) $\left(\frac{\pi^2}{a^3} \right)^{1/3}$ (d) $\frac{1}{a}$

Soln. Radius of fermi sphere is $k_F = \left(\frac{3\pi^2 N}{V} \right)^{1/3}$ for fcc solid $\frac{N}{V} = \frac{4}{a^3}$

$$k_F = \left(\frac{12\pi^2}{a^3} \right)^{1/3}$$

Correct option is (a)

73. The muon has mass $105 \text{ MeV}/c^2$ and mean lifetime $2.2 \mu\text{s}$ in its rest frame. The mean distance traversed by muon of energy $315 \text{ MeV}/c^2$ before decaying is approximately

(a) $3 \times 10^5 \text{ km}$ (b) 2.2 cm (c) $6.6 \mu\text{m}$ (d) 1.98 km

Soln. $m_0 = 105 \text{ MeV}/c^2, m = 315 \text{ MeV}/c^2$

$$\gamma = \frac{m}{m_0} = \frac{315}{105} = 3 \Rightarrow \frac{1}{\sqrt{1-v^2/c^2}} = 3 \Rightarrow v = \sqrt{8} \times 10^8 \text{ m/sec}$$

$$T(\text{dilated}) = \gamma T_0 = 3 \times 2.2 = 6.6 \mu\text{s}$$

$$\text{Distance traversed} = (\text{speed}) \times (\text{time}) = (\sqrt{8} \times 10^8 \text{ m/s}) (6.6 \times 10^{-6} \text{ s}) = (6.6 \times 2\sqrt{2}) \times 10^2 \text{ m} = 1.86 \text{ km}$$

Correct option is (d).

74. Consider the following particles: the proton p , the neutron n , the neutral pion π^0 and the delta resonance Δ^+ . When ordered in terms of decreasing lifetime, the correct arrangement is as follows:

(a) π^0, n, p, Δ^+ (b) p, n, Δ^+, π^0 (c) p, n, π^0, Δ^+ (d) Δ^+, n, π^0, p

Soln. The lifetime of $p \approx$ infinite (10^{26} years)

$$n \sim 1.01 \times 10^{-3} \text{ sec}$$

$$\pi^0 \sim 10^{-8} \text{ sec}$$

$$\Delta^+ \sim 10^{-10} \text{ sec}$$

Correct option is (c)

75. The single particle energy difference between the p-orbitals (i.e. $p_{3/2}$ and $p_{1/2}$) of the nucleus $^{114}_{50}\text{Sn}$ is 3 MeV. The energy difference between the states in its 1f orbital is

(a) -7 MeV (b) 7 MeV (c) 5 MeV (d) -5 MeV

Soln. The single particle energy difference

$$E_{l+\frac{1}{2}} - E_{l-\frac{1}{2}} = \left(\hbar^2 \left(l + \frac{1}{2} \right) \right) \text{ where 'C' is constant}$$

For p-orbitals, the energy difference,

$$E_{3/2} - E_{1/2} = \hbar^2 C \times \frac{3}{2} = 3 \text{ MeV}$$

$$\Rightarrow C = \frac{2}{\hbar^2} \text{ MeV} \quad (1)$$

The energy difference between the states in 1f orbital.

$$E_{7/2} - E_{5/2} = \hbar^2 C \times \frac{7}{2} \quad (2)$$

On substituting the value of 'C' from equation (1) into (2)

$$E_{7/2} - E_{5/2} = \hbar^2 \times \frac{2}{\hbar^2} \times \frac{7}{2} \text{ MeV} = 7 \text{ MeV}$$

Correct option is (b)



CSIR-UGC-NET/JRF- JUNE - 2013 PHYSICAL SCIENCES

Part-B

21. Two identical bosons of mass m are placed in a one-dimensional potential $V(x) = \frac{1}{2}m\omega^2 x^2$. The bosons interact via a weak potential, $V_{12} = V_0 \exp[-m\Omega(x_1 - x_2)^2 / 4\hbar]$ where x_1 and x_2 denote coordinates of the particles. Given that the ground state wavefunction of the harmonic oscillator is $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$. The ground state energy of the two-boson system, to the first order in V_0 , is:

- (a) $\hbar\omega + 2V_0$ (b) $\hbar\omega + \frac{V_0\Omega}{\omega}$
(c) $\hbar\omega + V_0\left(1 + \frac{\Omega}{2\omega}\right)^{-1/2}$ (d) $\hbar\omega + V_0\left(1 + \frac{\omega}{\Omega}\right)$

Soln. The Hamiltonian of the system of two identical Bosons

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2 + V_0 \exp\left[-\frac{m\Omega(x_1 - x_2)^2}{4\hbar}\right]$$

$$= H_0 + H_p$$

Where, $H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2$

and $H_p = V_{12} = V_0 \exp\left[-\frac{m\Omega(x_1 - x_2)^2}{4\hbar}\right]$

The unperturbed energy of system

$$E_n = \left(n_1 + \frac{1}{2}\right)\hbar\omega + \left(n_2 + \frac{1}{2}\right)\hbar\omega, \quad n_1 n_2 = 0, 1, 2, 3, \dots$$

The unperturbed ground state energy

$$E_0^{(0)} = \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega = \hbar\omega \quad \dots (1)$$

The first order correction in energy of ground state

$$E_0^1 = \langle \psi_0 | H_p | \psi_0 \rangle = \int \psi_0^* H_p \psi_0 d\tau$$

Where, $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_1^2 + x_2^2)}$

Then, $E_0^1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_1^2 + x_2^2)} V_0 e^{-\frac{m\Omega(x_1 - x_2)^2}{4\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_1^2 + x_2^2)} dx_1 dx_2$

$$= V_0 \left(\frac{m\omega}{\pi\hbar} \right) \int_{-\infty}^{\infty} e^{-\left(\frac{m\omega}{\hbar} + \frac{m\Omega}{4\hbar}\right)x_2^2} \left[\int_{-\infty}^{\infty} e^{-\left[\left(\frac{m\omega}{\hbar} + \frac{m\Omega}{4\hbar}\right)x_1^2 - \left(\frac{m\Omega}{2\hbar}x_2\right)x_1\right]} dx_1 \right] dx_2 \quad \dots (2)$$

Note: $\int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} \exp\left[\frac{\beta^2}{4\alpha}\right]$

The value of the integral.

$$\int_{-\infty}^{+\infty} e^{-\left[\left(\frac{m\omega}{\hbar} + \frac{m\Omega}{4\hbar}\right)x_1^2 - \left(\frac{m\Omega}{2\hbar}x_2\right)x_1\right]} dx_1 = \left(\frac{\pi}{\frac{m\omega}{\hbar} + \frac{m\Omega}{4\hbar}} \right)^{1/2} \exp \left[\frac{\left(\frac{m\Omega x_2}{2\hbar} \right)^2}{\left(\frac{4m\omega}{\hbar} + \frac{m\Omega}{\hbar} \right)} \right]$$

$$= \left(\frac{\pi}{\frac{m\Omega}{\hbar} + \frac{m\Omega}{4\hbar}} \right)^{1/2} \exp \left[\frac{m^2 \Omega^2 x_2^2}{4\hbar(4m\omega + m\Omega)} \right] \quad \dots (3)$$

On substituting the value of the integral from equation (3) into equation (2), we get

$$E_0^1 = V_0 \left(\frac{m\omega}{\pi\hbar} \right) \left(\frac{\pi}{\frac{m\omega}{\hbar} \left(1 + \frac{\Omega}{4\omega} \right)} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left[- \left\{ \frac{m\omega}{\hbar} + \frac{m\Omega}{4\hbar} - \frac{m^2 \Omega^2}{4\hbar(4m\omega + m\Omega)} \right\} x_2^2 \right] dx_2 \quad \dots (4)$$

Consider:

$$\begin{aligned} \frac{m\omega}{\hbar} + \frac{m\Omega}{4\hbar} - \frac{m^2 \Omega^2}{4\hbar(4m\omega + m\Omega)} &= \frac{4m\omega(4m\omega + m\Omega) + m\Omega(4m\omega + m\Omega) - m^2 \Omega^2}{4\hbar(4m\omega + m\Omega)} \\ &= \frac{16m^2 \omega^2 + 8m^2 \omega \Omega}{4\hbar(4m\omega + m\Omega)} = \frac{16m^2 \omega^2 \left(1 + \frac{\Omega}{2\omega} \right)}{16\hbar m \omega \left(1 + \frac{\Omega}{4\omega} \right)} = \frac{m\omega}{\hbar} \left(\frac{1 + \frac{\Omega}{2\omega}}{1 + \frac{\Omega}{4\omega}} \right) \end{aligned}$$

Therefore,
$$E_0^1 = V_0 \left(\frac{m\omega}{\pi\hbar} \right) \left(\frac{\pi}{\frac{m\omega}{\hbar} \left(1 + \frac{\Omega}{4\omega} \right)} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left[- \left\{ \frac{m\omega}{\hbar} \left(\frac{1 + \frac{\Omega}{2\omega}}{1 + \frac{\Omega}{4\omega}} \right) \right\} x_2^2 \right] dx_2$$

$$= V_0 \left(\frac{m\omega}{\pi\hbar} \right) \left(\frac{\pi}{\frac{m\omega}{\hbar} \left(1 + \frac{\Omega}{4\omega} \right)} \right)^{1/2} \frac{\Gamma \frac{1}{2}}{\left[\frac{m\omega}{\hbar} \left(\frac{1 + \Omega/2\omega}{1 + \Omega/4\omega} \right) \right]^{1/2}} = V_0 \left(1 + \frac{\Omega}{2\omega} \right)^{-1/2}$$

Then ground state energy corrected to first order

$$E'_0 = E_0^{(0)} + E_0^1 = \hbar\omega + V_0 \left(1 + \frac{\Omega}{2\omega}\right)^{-1/2}$$

Correct answer is (c)

22. Ten grams of ice at 0°C is added to a beaker containing 30 grams of water at 25°C . What is the final temperature of the system when it comes to thermal equilibrium? (The specific heat of water is $1\text{ cal/gm}^\circ\text{C}$ and latent heat of melting of ice is 80 cal/gm)
 (a) 0°C (b) 7.5°C (c) 12.5°C (d) -1.25°C

Soln. The heat flows spontaneously from hot body to cold body. It means heat will be flowed from water to ice.

If water temperature changes from 25°C to 0°C , then heat given by the water = $30 \times 1 \times 25 = 750\text{ cal}$.

To convert ice into water at 0°C completely the heat required = $mL = 10 \times 80 = 800\text{ Cal}$

Heat given < heat required

So ice is not converted into water completely so the final temperature is 0°C .

Correct answer is (a)

23. A vessel has two compartments of volume V_1 and V_2 , containing an ideal gas at pressures p_1 and p_2 , and temperatures T_1 and T_2 respectively. If the wall separating the compartments is removed, the resulting equilibrium temperature will be

- (a) $\frac{p_1 T_1 + p_2 T_2}{p_1 + p_2}$ (b) $\frac{V_1 T_1 + V_2 T_2}{V_1 + V_2}$
 (c) $\frac{p_1 V_1 + p_2 V_2}{(p_1 V_1 / T_1) + (p_2 V_2 / T_2)}$ (d) $(T_1 T_2)^{1/2}$

Soln.

V_1	V_2
p_1	p_2
T_1	T_2

If $T_2 > T_1$, on removing the wall the heat will flow from ideal gas at T_2 towards ideal gas at T_1 . Heat given = Heat taken (Calorimetry principle).

Let equilibrium temperature be T_f then

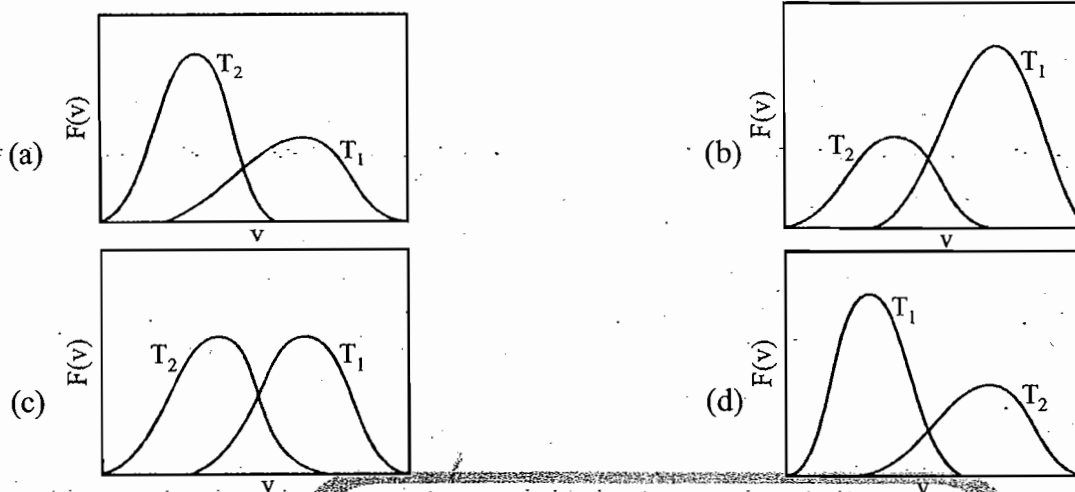
$$Cp_2(T_2 - T_f) = Cp_1(T_f - T_1), \quad n_2 R(T_2 - T_f) = n_1 R(T_f - T_1)$$

where C_p = heat capacity at constant pressure

$$T_f = \frac{(n_1 T_1 + n_2 T_2)}{n_1 + n_2}, \quad T_f = \frac{p_1 V_1 + p_2 V_2}{\left(\frac{p_1 V_1}{T_1}\right) + \left(\frac{p_2 V_2}{T_2}\right)} \quad [p_1 V_1 = n_1 R T_1, p_2 V_2 = n_2 R T_2]$$

Correct answer is (c)

24. For temperature $T_1 > T_2$, the qualitative temperature dependence of the probability distribution $F(v)$ of the speed v of a molecule in three dimensions is correctly represented by the following figure:

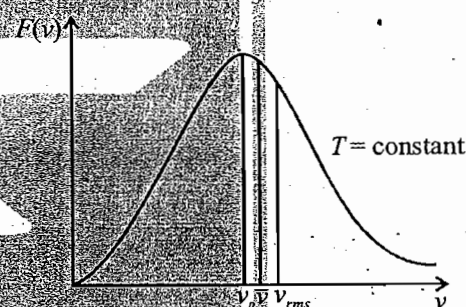


Soln. For N-particles system in thermal equilibrium, the number of molecules having speed between v to $v+dv$ is given by

$$F(v)dv = 4\pi N \left(\frac{m}{2\pi KT} \right)^{3/2} e^{-\frac{mv^2}{2KT}} v^2 dv$$

$$F(v) = 4\pi N \left(\frac{m}{2\pi KT} \right)^{3/2} e^{-\frac{mv^2}{2KT}} v^2 \quad \text{and} \quad \int_0^\infty F(v)dv = N$$

$$\text{and } \bar{v} = \sqrt{\frac{8KT}{\pi m}}, v_p = \sqrt{\frac{2KT}{m}}, v_{rms} = \sqrt{\frac{3KT}{m}}$$



If we increase temperature, then $\bar{v}$, v_p and v_{rms} all will increase and particles have more values of velocities. More particles will distribute in large velocity region but total number of particles in the system will be constant (N). In $F(v)$ vs v plot area under the curve remain constant. So an increasing temperature peak will decrease and shifted towards higher velocity side.

Correct answer is (a)

25. A system of non-interacting spin-1/2 charged particles are placed in an external magnetic field. At low temperature T , the leading behavior of the excess energy above the ground state energy, depends on T as: (c is a constant)
- (a) cT (b) cT^3 (c) $e^{-c/T}$ (d) c (is independent of T)
26. The acceleration due to gravity g is determined by measuring the time period T and the length L of a simple pendulum. If the uncertainties in the measurements of T and L are ΔT and ΔL respectively, the fractional error $\Delta g/g$ in measuring g is best approximated by

(a) $\frac{|\Delta L|}{L} + \frac{|\Delta T|}{T}$

(b) $\frac{|\Delta L|}{L} + \frac{2|\Delta T|}{T}$

(c) $\sqrt{\frac{|\Delta L|^2}{L^2} + \frac{|\Delta T|^2}{T^2}}$

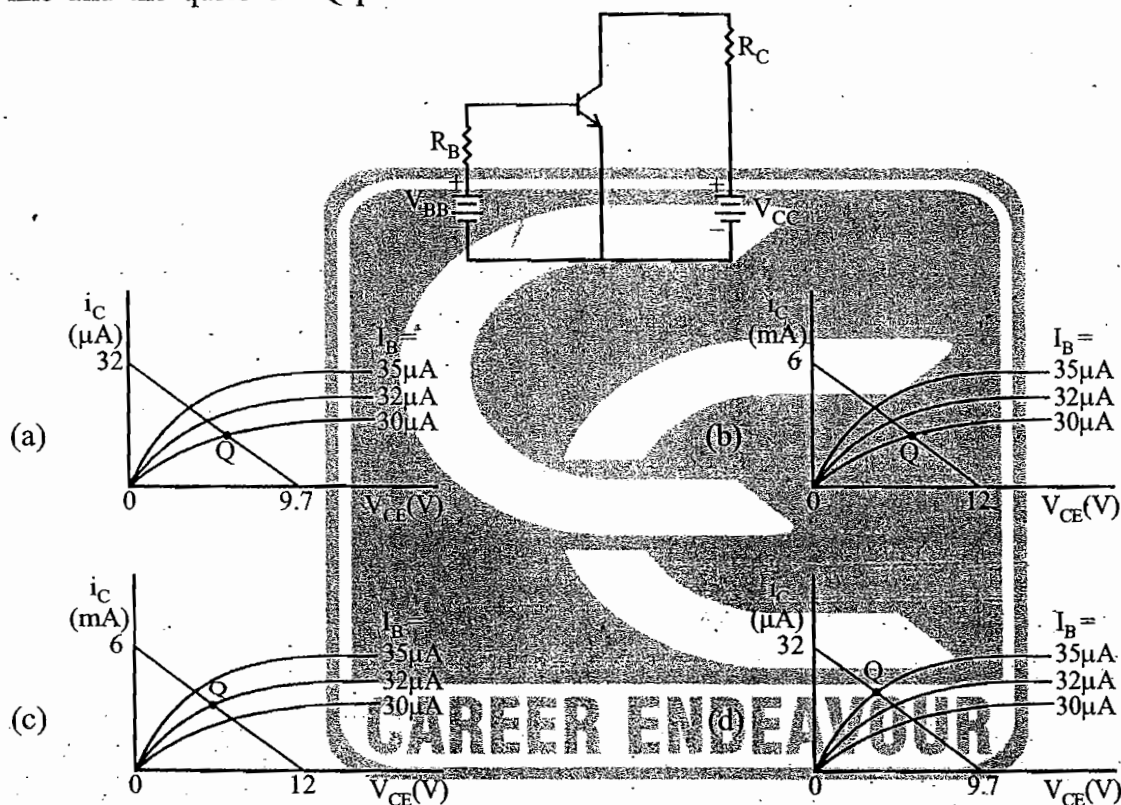
(d) $\sqrt{\left(\frac{\Delta L}{L}\right)^2 + \left(\frac{2\Delta T}{T}\right)^2}$

Soln. $T = 2\pi\sqrt{L/g} \Rightarrow g = \frac{4\pi^2 L}{T^2}$

relative error in g is given by $\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta L}{L}\right)^2 + \left(\frac{2\Delta T}{T}\right)^2}$

Correct answer is (d)

27. A silicon transistor with built-in voltage $0.7V$ is used in the circuit shown, with $V_{BB} = 9.7V$, $R_B = 300k\Omega$, $V_{CC} = 12V$ and $R_C = 2k\Omega$. Which of the following figures correctly represents the load line and the quiescent Q point?



Soln. For DC load line

$$V_{CE} = V_{CC}, V_{CE} = 12V, I_C = \frac{V_{CC}}{R_C} = 6mA$$

Calculation of I_C

$$-V_{BB} + I_B \times R_B + V_{BE} = 0 \Rightarrow I_B \times R_B = 90V$$

$$\Rightarrow I_B = \frac{9}{300K} = \frac{3}{100}mA = 0.03mA$$

Correct answer is (b)

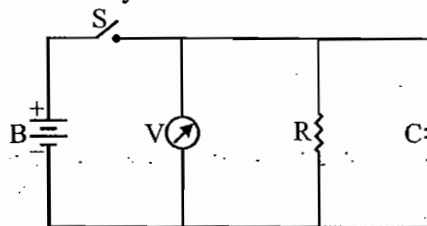
28. If the analog input to an 8-bit successive approximation ADC is increased from $1.0V$ to $2.0V$, then the conversion time will

- (a) remain unchanged (b) double
(c) decrease to half its original value (d) increase four times

Soln. Successive approximation converter are also called as uniform, uniform converter. Hence remain unchanged.

Correct answer is (a)

29. The insulation resistance R of an insulated cable is measured by connecting it in parallel with a capacitor C , a voltmeter, and battery B as shown. The voltage across the cable dropped from 150V to 15V, 1000 seconds after the switch S is closed. If the capacitance of the cable is $5\mu\text{F}$ then its insulation resistance is approximately



- (a) $10^9\Omega$ (b) $10^8\Omega$ (c) $10^7\Omega$ (d) $10^6\Omega$

Soln. $C = \frac{q}{V} \Rightarrow q = CV = 15 \times 5\mu\text{f} \Rightarrow q = 75\mu\text{C}$

So, $I = \frac{q}{T} = \frac{75\mu\text{C}}{1000} = 75\text{ nA} \Rightarrow R = \frac{V}{I} = \frac{15}{75\text{ nA}} = 2 \times 10^8\Omega \approx 10^8\Omega$

Correct answer is (b)

30. The approximation $\cos \theta \approx 1$ is valid up to 3 decimal places as long as $|\theta|$ is less than: (take $180^\circ/\pi = 57.29^\circ$).
- (a) 1.28° (b) 1.81° (c) 3.28° (d) 4.01°

Soln. $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \approx 1 - \frac{\theta^2}{2!}$ (for small θ)

Now, for $\theta = 1.81^\circ \approx \frac{\pi}{100} = 0.0314$ (approx.)

$\Rightarrow \cos \theta \approx 1 - \frac{(0.0314)^2}{2!} = 1 - (0.00049..) = 1$ (upto 3 decimal places)

Correct answer is (b)

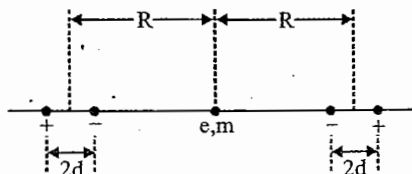
31. The area of a disc in its rest frame S is equal to 1 (in some units). The disc will appear distorted to an observer O moving with a speed u with respect to S along the plane of the disc. The area of the disc measured in the rest frame of the observer O is (c is the speed of light in vacuum)

- (a) $\left(1 - \frac{u^2}{c^2}\right)^{1/2}$ (b) $\left(1 - \frac{u^2}{c^2}\right)^{-1/2}$ (c) $\left(1 - \frac{u^2}{c^2}\right)$ (d) $\left(1 - \frac{u^2}{c^2}\right)^{-1}$

Soln. Using $A = A_0 \sqrt{1 - \frac{u^2}{c^2}}$

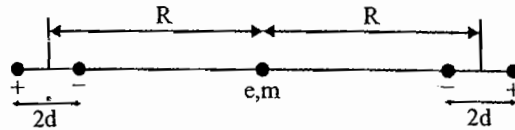
Correct answer is (a)

32. A particle of charge e and mass m is located at the midpoint of the line joining two mixed collinear dipoles with unit charges as shown in the figure. (The particle is constrained to move only along the line joining the dipoles). Assuming that the length of the dipoles is much shorter than their separation, the natural frequency of oscillation of the particle is



- (a) $\sqrt{\frac{6e^2 R^2}{\pi \epsilon_0 m d^5}}$ (b) $\sqrt{\frac{6e^2 R}{\pi \epsilon_0 m d^4}}$ (c) $\sqrt{\frac{6e^2 d^2}{\pi \epsilon_0 m R^5}}$ (d) $\sqrt{\frac{6e^2 d}{\pi \epsilon_0 m R^4}}$

soln. The dipole moment of each dipole is $p = e2d$.



Electric field of two dipoles directed opposite to each other at position of e . Hence, if charge is displaced by a small distance x towards right. Then force on charge e is given by

$$F = \left(\frac{2p_1}{4\pi\epsilon_0(R+x)^3} - \frac{2p_2}{4\pi\epsilon_0(R-x)^3} \right) e$$

$$F = \frac{4e^2 d}{4\pi\epsilon_0} \left[\frac{1}{(R+x)^3} - \frac{1}{(R-x)^3} \right] = \frac{e^2 d [-x^3 - 6R^2 x]}{\pi\epsilon_0 (R+x)^3 (R-x)^3}$$

But $x \ll R$ hence

$$F = -\frac{6e^2 d R^2 x}{\pi\epsilon_0 R^6 R^4} = -\frac{6e^2 d}{\pi\epsilon_0 R^4} x \quad (1)$$

Again $F = -kx$ (2)

By (1) and (2), $K = \frac{6e^2 d}{\pi\epsilon_0 R^4}$

Frequency of oscillation

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{6e^2 d}{\pi\epsilon_0 m R^4}}$$

Correct answer is (d)

13.

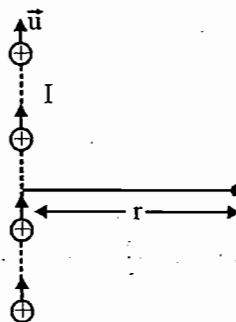
A current I is created by a narrow beam of protons moving in vacuum with constant velocity $\vec{u}$. The direction and magnitude, respectively, of the Poynting vector $\vec{S}$ outside the beam at a radial distance r (much larger than the width of the beam) from the axis, are

- (a) $\vec{S} \perp \vec{u}$ and $|\vec{S}| = \frac{I^2}{4\pi^2 \epsilon_0 |\vec{u}| r^2}$ (b) $\vec{S} \parallel (-\vec{u})$ and $|\vec{S}| = \frac{I^2}{4\pi^2 \epsilon_0 |\vec{u}| r^4}$
- (c) $\vec{S} \parallel \vec{u}$ and $|\vec{S}| = \frac{I^2}{4\pi^2 \epsilon_0 |\vec{u}| r^2}$ (d) $\vec{S} \parallel \vec{u}$ and $|\vec{S}| = \frac{I^2}{4\pi^2 \epsilon_0 |\vec{u}| r^4}$

soln. Let charge per unit length is λ , hence,

$$I = \lambda u \quad \dots (1)$$

the magnetic field at a distance r is $\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$... (2)



Electric field at distance r is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{s}, \text{ but from equation (1)}$$

We get
$$E = \frac{I}{2\pi\epsilon_0 u r} \hat{s} \quad (3)$$

Hence, Poynting vector
$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$$\Rightarrow \vec{S} = \frac{I^2}{4\pi^2 \epsilon_0 u r^2} \hat{z} \quad (4)$$

$\hat{z}$ is direction of $\vec{u}$ hence **Correct answer is (c)**

34. If the electric and magnetic fields are unchanged when the vector potential $\vec{A}$ changes (in suitable units) according to $\vec{A} \rightarrow \vec{A} + \vec{r}$, where $\vec{r} = r(t)\hat{r}$, then the scalar potential ϕ must simultaneously change to

- (a) $\phi - r$ (b) $\phi + r$ (c) $\phi - \partial r / \partial t$ (d) $\phi + \partial r / \partial t$

Soln. $\vec{A} \rightarrow \vec{A} + \vec{r}$

If electric and magnetic field needs to be unchanged. Then vector and scalar potentials must satisfy the conditions,

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad \dots (2)$$

$$\text{and } \phi' = \phi - \frac{\partial \lambda}{\partial t} \quad \dots (3)$$

where λ is gauge function

Comparing (1) and (2), we get

$$\vec{\nabla}\lambda = \hat{r} \Rightarrow \lambda = r(t)$$

Hence,
$$\phi' = \phi - \frac{\partial \lambda}{\partial t} = \phi - \frac{\partial r(t)}{\partial t} \quad \dots (4)$$

Correct answer is (c)

35. Consider an axially symmetric static charge distribution of the form,

$\rho = \rho_0 \left(\frac{r_0}{r} \right)^2 e^{-r/r_0} \cos^2 \phi$. The radial component of the dipole moment due to this charge distribution is

- (a) $2\pi\rho_0 r_0^4$ (b) $\pi\rho_0 r_0^4$ (c) $\rho_0 r_0^4$ (d) $\pi\rho_0 r_0^4 / 2$

Soln. $\rho = \rho_0 \left(\frac{r_0}{r} \right)^2 e^{-r/r_0} \cos^2 \phi$

radial component of the dipole moment is given by

$$p = \int_V \rho \vec{r} d\tau = \int_0^\infty \int_0^\pi \int_0^{2\pi} \rho_0 \left(\frac{r_0}{r} \right)^2 e^{-r/r_0} \cos^2 \phi \cdot r \cdot r^2 \sin \theta d\theta d\phi dr$$

$$= 2r_0^2 \rho_0 \int_0^\infty r e^{-r/r_0} dr \int_0^\pi \cos^2 \phi d\phi = 2r_0^2 \rho_0 \left[\frac{1}{\left(\frac{1}{r_0} \right)^2} \right] \int_0^\pi \left[\frac{1 + \cos 2\phi}{2} \right] d\phi = 2\pi\rho_0 r_0^4$$

Correct answer is (a)

36. In a basis in which the z-component S_z of the spin is diagonal, an electron is in a spin state

$$\psi = \begin{pmatrix} (1+i)/\sqrt{6} \\ \sqrt{2/3} \end{pmatrix}$$

The probabilities that a measurement of S_z will yield the values $\hbar/2$ and $-\hbar/2$ are, respectively.

- (a) 1/2 and 1/2 (b) 2/3 and 1/3 (c) 1/4 and 3/4 (d) 1/3 and 2/3

Soln. The eigenfunctions of S_z in a basis in which S_z is diagonal are

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } \frac{\hbar}{2} \text{ eigenvalue}$$

and $|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for } -\frac{\hbar}{2} \text{ eigenvalue}$

The given state can be written as linear superposition of $|\phi_1\rangle$ and $|\phi_2\rangle$ as

$$\begin{pmatrix} (1+i)/\sqrt{6} \\ \sqrt{2/3} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{(1+i)}{\sqrt{6}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{2}{3}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The probability of yielding the value $+\frac{\hbar}{2}$

$$= \left| \frac{1+i}{\sqrt{6}} \right|^2 = \frac{(1+i)(1-i)}{6} = \frac{2}{6} = \frac{1}{3}$$

and the probability of yielding the value $-\frac{\hbar}{2}$ $= \left| \sqrt{\frac{2}{3}} \right|^2 = \frac{2}{3}$

Correct answer is (d)

37. Consider the normalized state $|\psi\rangle$ of a particle in a one-dimensional harmonic oscillator:
 $|\psi\rangle = b_1|0\rangle + b_2|1\rangle$ where $|0\rangle$ and $|1\rangle$ denote the ground and first excited states respectively, and b_1 and b_2 are real constants. The expectation value of the displacement x in the state $|\psi\rangle$ will be a minimum when

(1) $b_2 = 0, (b_1 = 1)$ (2) $b_2 = \frac{1}{\sqrt{2}} b_1$ (3) $b_2 = \frac{1}{2} b_1$ (4) $b_2 = b_1$

Soln. Given : Normalized state $|\psi\rangle$ of a particle in a one-dimensional harmonic oscillator is

$$|\psi\rangle = b_1|0\rangle + b_2|1\rangle$$

Normalization condition : $b_1^2 + b_2^2 = 1$

Here $|0\rangle \rightarrow$ ground state

$|1\rangle \rightarrow$ first excited state

Expectation value of the displacement x in the state $|\psi\rangle = \langle\psi|x|\psi\rangle$

$$= b_1^2 \langle 0|x|0\rangle + b_2^2 \langle 1|x|1\rangle + b_1 b_2 \langle 0|x|1\rangle + b_2 b_1 \langle 1|x|0\rangle$$

Since, x is an odd function, therefore first two terms of the above expression will be zero.

$$P = \langle\psi|x|\psi\rangle = b_1 b_2 \langle 0|x|1\rangle + b_2 b_1 \langle 1|x|0\rangle$$

$$= b_1 b_2 \sqrt{\frac{\hbar}{2m\omega}} [\langle 0|\hat{a} + \hat{a}^\dagger|1\rangle + \langle 1|\hat{a} + \hat{a}^\dagger|0\rangle]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} b_1 b_2 [\langle 0|\sqrt{1}|0\rangle + \langle 0|\sqrt{2}|2\rangle + 0 + \langle 1|\sqrt{1}|1\rangle]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} b_1 b_2 (1+1) = 2b_1 b_2 \sqrt{\frac{\hbar}{2m\omega}}$$

Now P will be minimum if $b_1 b_2$ is minimum.

$$\text{Therefore, } b_1 b_2 = \frac{1}{4}(4b_1 b_2) = \frac{1}{4}[(b_1 + b_2)^2 - (b_1 - b_2)^2] = S \text{ (say)}$$

So, $b_1 b_2$ will be minimum if $b_2 = 0$ or $b_1 = 0$

Case 1: $b_1 = 0 \Rightarrow b_2 = 1$

Case 2: $b_2 = 0 \Rightarrow b_1 = 1$ [Due to normalization condition $b_1^2 + b_2^2 = 1$]

So, correct option should be (a)

In the answer key correct option is given to be (d) i.e. $b_1 = b_2$

If $b_1 = b_2 \Rightarrow b_1 b_2$ will be maximum and its maximum value will be b_1^2 or b_2^2 then, $\langle\psi|x|\psi\rangle$

will be maximum and its maximum value will be $2b_1^2 \sqrt{\frac{\hbar}{2m\omega}}$ or $2b_2^2 \sqrt{\frac{\hbar}{2m\omega}}$

38. A muon (μ^-) from cosmic rays is trapped by a proton to form a hydrogen-like atom. Given that

a muon is approximately 200 times heavier than an electron, the longest wavelength of the spectral line (in the analogue of the Lyman series) of such an atom will be

- (a) 5.62 Å (b) 6.67 Å (c) 3.75 Å (d) 13.3 Å

Soln. Reduced mass of muonic-hydrogen is given by

$$\mu_{\mu} = \frac{200m_e \cdot m_p}{(200m_e + m_p)} \approx 181m_e \quad \dots (1)$$

Longest wavelength of Lyman series corresponds to $n_i = 2 \rightarrow n_f = 1$ transition

$$\frac{1}{\lambda} = R_M \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

$$\Rightarrow \frac{1}{\lambda} = \frac{3}{4} R_M$$

$$\Rightarrow \lambda = \frac{4}{3R_M}$$

$$\frac{(\lambda)_{\mu^-}}{(\lambda)_H} = \frac{(R_M)_H}{(R_M)_{\mu^-}} = \frac{(\mu)_H}{(\mu)_{\mu^-}} = \frac{1}{186}$$

$$\text{So, } (\lambda)_{\mu^-} = \frac{1}{186} (\lambda)_H = 6.67 \text{ Å}$$

Correct answer is (b)

39. The un-normalized wavefunction of a particle in a spherically symmetric potential is given by $\psi(\vec{r}) = zf(r)$ where $f(r)$ is a function of the radial variable r . The eigenvalue of the operator $\vec{L}^2$ (namely the square of the orbital angular momentum) is

- (a) $\hbar^2/4$ (b) $\hbar^2/2$ (c) $\hbar^2$ (d) $2\hbar^2$

Soln. The un-normalized wavefunction $\psi(\vec{r}) = Z f(r)$

In spherical polar co-ordinate system

$$\psi(\vec{r}) = r \cos \theta f(r)$$

$\psi(\vec{r})$ has only power of $\cos \theta$ so the orbital quantum number $\ell = 1$

The eigenvalue of $L^2 = \ell(\ell+1)\hbar^2 = 2\hbar^2$

Correct answer is (d)

40. Given that $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{-t^2+2tx}$ the value of $H_4(0)$ is

- (a) 12 (b) 6 (c) 24 (d) -6

Soln. The right hand side of the given relation is the generating function for the Hermite polynomial

$H_n(x)$. Hermite polynomial of order '4' is $H_4(x) = 16x^4 - 48x^2 + 12$. Therefore, the value of

$$H_4(0) = 12$$

Correct answer is (a)

41. A unit vector $\hat{n}$ on the xy-plane is at an angle of 120° with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60° if

(a) $b = \sqrt{3}a/2$ (b) $b = 2a/\sqrt{3}$ (c) $b = a/2$ (d) $b = a$

Soln. Since, the unit vector makes an angle 120° with x-axis, then it makes an angle 30° with y-axis.

Therefore, $\hat{n} = \cos 120^\circ \hat{i} + \cos 30^\circ \hat{j} = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$

So, $\vec{u}$ and $\vec{v}$ can be written as, $\vec{u} = \left(a - \frac{b}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}b\hat{j}$ and $\vec{v} = \left(b - \frac{a}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}a\hat{j}$

Since, the angle between $\vec{u}$ and $\vec{v}$ is 60° , therefore $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos 60^\circ$

$$\Rightarrow \left(a - \frac{b}{2}\right)\left(b - \frac{a}{2}\right) + \frac{3}{4}ab = \sqrt{\left(a - \frac{b}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2} \sqrt{\left(b - \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2} \cos 60^\circ$$

$$\Rightarrow 2ab - \frac{b^2}{2} - \frac{a^2}{2} = (a^2 + b^2 - ab) \cdot \frac{1}{2} \Rightarrow 4ab - b^2 - a^2 = a^2 + b^2 - ab$$

$$\Rightarrow 2a^2 - 5ab + 2b^2 = 0 \Rightarrow (2a - b)(2b - a) = 0 \Rightarrow a = \frac{b}{2} \text{ or } b = \frac{a}{2}$$

Correct answer is (c)

42. With $z = x + iy$, which of the following functions $f(x, y)$ is NOT a (complex) analytic function of z ?

(a) $f(x, y) = (x + iy - 8)^3 (4 + x^2 - y^2 + 2ixy)^7$

(b) $f(x, y) = (x + iy)^7 (1 - x - iy)^3$

(c) $f(x, y) = (x^2 - y^2 + 2ixy - 3)^5$

(d) $f(x, y) = (1 - x + iy)^4 (2 + x + iy)^6$

Soln. Any complex function $f(x, y)$ will be not analytic in nature if it contains $\bar{z}$ term. Given options can be written as

(1) $f(x, y) = (z - 8)^3 (4 + z^2)^7$,

(2) $f(x, y) = z^7 (1 - z)^3$

(3) $f(x, y) = (z^2 - 3)^5$,

(4) $f(x, y) = (1 - \bar{z})^4 (2 + z)$

Only option (4) contains $\bar{z}$ term. Therefore, option (4) will not be analytic in nature.

Correct answer is (d)

43. A planet of mass m and an angular momentum L moves in a circular orbit in a potential, $V(r) = -k/r$, where k is a constant. If it is slightly perturbed radially, the angular frequency of radial oscillations is

(a) $mk^2 / \sqrt{2} L^3$

(b) mk^2 / L^3

(c) $\sqrt{2}mk^2 / L^3$

(d) $\sqrt{3}mk^2 / L^3$

Soln. Total energy of planet is written as

$E = T + V(r)$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{K}{r}}_{V_{eff}}$$

$\therefore$ effective potential is $V_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{k}{r}$

$\left. \frac{dV_{\text{eff}}}{dr} \right|_{r=r_0} = 0$ gives radius of stationary orbit.

$$\Rightarrow -\frac{L^2}{mr_0^3} + \frac{K}{r_0^2} = 0 \Rightarrow r_0 = \frac{L^2}{mK}$$

Force constant of oscillation

$$K = \left. \frac{d^2V_{\text{eff}}}{dr^2} \right|_{r=r_0} = \frac{3L^2}{mr_0^4} - \frac{2K}{r_0^3} = \frac{1}{r_0^3} \left[\frac{3L^2}{m r_0} - 2K \right] = \frac{1}{r_0^3} \left[\frac{3L^2}{m \cdot \frac{L^2}{mK}} - 2K \right] = K/r_0^3 = \frac{m^3 K^4}{L^6}$$

frequency of oscillation $\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{m^3 K^4}{m L^6}} = \frac{m K^2}{L^3}$

Correct answer is (b)

44. The Lagrangian of a particle of mass m moving in one dimensions is given by

$$L = \frac{1}{2} m \dot{x}^2 - bx$$

where b is a positive constant. The coordinate of the particle $x(t)$ at time t is given by: (in the following c_1 and c_2 are constants)

- (a) $-\frac{b}{2m} t^2 + c_1 t + c_2$ (b) $c_1 t + c_2$
(c) $c_1 \cos\left(\frac{bt}{m}\right) + c_2 \sin\left(\frac{bt}{m}\right)$ (d) $c_1 \cosh\left(\frac{bt}{m}\right) + c_2 \sinh\left(\frac{bt}{m}\right)$

Soln. $L = \frac{1}{2} m \dot{x}^2 - bx = T - V$

$$\therefore V = bx$$

This is same as uniform gravitational potential. Therefore, solution must be of the form.

$$S = S_0 + ut + \frac{1}{2} at^2$$

Correct answer is (a)

45. A uniform cylinder of radius r and length ℓ , and a uniform sphere of radius R are released on an inclined plane when their centres of mass are at the same height. If they roll down without slipping, and if the sphere reaches the bottom of the plane with a speed V , then the speed of the cylinder when it reaches the bottom is:

- (a) $V \sqrt{\frac{14r\ell}{15R^2}}$ (b) $4V \sqrt{\frac{r}{15R}}$ (c) $\frac{4V}{\sqrt{15}}$ (d) $V \sqrt{\frac{14}{15}}$

Soln. On an inclined plane acceleration of a rolling object is given as $a = \frac{g \sin \theta}{1 + \frac{I}{mr^2}}$

Therefore, for solid cylinder $I = mr^2/2$

$$a_c = \frac{2g \sin \theta}{3}$$

for solid sphere $I = \frac{2}{5} mr^2$

$$\therefore a_s = \frac{5g \sin \theta}{7}$$

On travelling a distance speed is given by $v^2 = 2as$

$$\left(\frac{V_c}{V_s}\right)^2 = \frac{a_c}{a_s} = \frac{14}{15}$$

given $V_s = V$;

$$\frac{V_c}{V} = \sqrt{\frac{14}{15}} \Rightarrow V_c = V \sqrt{\frac{14}{15}}$$

Correct answer is (d)

Part-C

46. The components of a vector potential $A_\mu \equiv (A_0, A_1, A_2, A_3)$ are given by $A_\mu = k(-xyz, yzt, zxt, xyt)$ where k is a constant. The three components of the electric field are
 (a) $k(yz, zx, xy)$ (b) $k(x, y, z)$ (c) $(0, 0, 0)$ (d) $k(xt, yt, zt)$

Soln. Vector potential in four vector representation is

$$A_\mu = k(-xyz, yzt, zxt, xyt)$$

It is equivalent to

$$A_\mu \equiv (\phi, A_x, A_y, A_z)$$

$$\text{Hence, } \phi = -xyz, A_x = yzt, A_y = zxt, A_z = xyt \Rightarrow A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

Electric field is given by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi, \text{ but } \vec{\nabla} \phi = -yz\hat{i} - xz\hat{j} - xy\hat{k}$$

$$\vec{E} = 0 \Rightarrow E_x = 0, E_y = 0, E_z = 0 \Rightarrow (0, 0, 0)$$

Correct answer is (c)

47. In the Born approximation, the scattering amplitude $f(\theta)$ for the Yukawa potential $V(r) = \frac{\beta e^{-\mu r}}{r}$

is given by: (in the following $b = 2k \sin \frac{\theta}{2}$, $E = \hbar^2 k^2 / 2m$)

$$(a) -\frac{2m\beta}{\hbar^2(\mu^2 + b^2)^2} \quad (b) -\frac{2m\beta}{\hbar^2(\mu^2 + b^2)} \quad (c) -\frac{2m\beta}{\hbar^2\sqrt{\mu^2 + b^2}} \quad (d) -\frac{2m\beta}{\hbar^2(\mu^2 + b^2)^3}$$

Soln. In the Born approximation, the scattering amplitude

$$\begin{aligned}
 f(\theta) &= \frac{-2m}{\hbar^2 k} \int_0^\infty r \sin kr V(r) dr & \text{where } k = 2k \sin \frac{\theta}{2} = b \\
 &= \frac{-2m}{\hbar^2 b} \int_0^\infty r \sin br V(r) dr = \frac{-2m}{\hbar^2 b} \int_0^\infty r \left(\frac{e^{ibr} - e^{-ibr}}{2i} \right) \frac{\beta e^{-\mu r}}{r} dr \\
 &= \frac{-m\beta}{i\hbar^2 b} \left[\int_0^\infty e^{ibr-\mu r} dr - \int_0^\infty e^{-ibr-\mu r} dr \right] \\
 &= -\frac{m\beta}{i\hbar^2 b} \left[\int_0^\infty e^{-(ib+\mu)r} dr - \int_0^\infty e^{-(ib-\mu)r} dr \right] \\
 &= \frac{-m\beta}{i\hbar^2 b} \left[\frac{1}{\mu-ib} - \frac{1}{\mu+ib} \right] = \frac{-m\beta}{i\hbar^2 b} \left[\frac{\mu+ib-\mu+ib}{\mu^2+b^2} \right] = \frac{-2m\beta}{\hbar^2(\mu^2+b^2)}
 \end{aligned}$$

Correct answer is (b)

48. If ψ_{nlm} denotes the eigenfunction of the Hamiltonian with a potential $V = V(r)$ then the expectation value of the operator $L_x^2 + L_y^2$ in the state $\psi = \frac{1}{5}[3\psi_{211} + \psi_{210} - \sqrt{15}\psi_{21-1}]$ is

- (a) $39\hbar^2/25$ (b) $13\hbar^2/25$ (c) $2\hbar^2$ (d) $26\hbar^2/25$

Soln.

$$\psi = \frac{1}{5}[3\psi_{211} + \psi_{210} - \sqrt{15}\psi_{21-1}]$$

$$|\psi\rangle = \frac{1}{5}[3|\psi_{211}\rangle + |\psi_{210}\rangle - \sqrt{15}|\psi_{21-1}\rangle]$$

$$\langle\psi| = \frac{1}{5}[3\langle\psi_{211}| + \langle\psi_{210}| - \sqrt{15}\langle\psi_{21-1}|]$$

The expectation value of the operator

$$\langle(L_x^2 + L_y^2)\rangle = \langle L_x^2 + L_y^2 + L_z^2 - L_z^2 \rangle = \langle L^2 - L_z^2 \rangle = \langle L^2 \rangle - \langle L_z^2 \rangle \quad \dots (1)$$

$$\langle L^2 \rangle = \langle \psi | L^2 | \psi \rangle$$

$$= \frac{1}{5}[\langle\psi_{211}|3 + \langle\psi_{210}| - \sqrt{15}\langle\psi_{21-1}|] \frac{1}{5}[3L^2|\psi_{211}\rangle + L^2|\psi_{210}\rangle - \sqrt{15}L^2|\psi_{21-1}\rangle]$$

$$= \frac{1}{25}[\langle\psi_{211}|3 + \langle\psi_{210}| - \sqrt{15}\langle\psi_{21-1}|][3 \times 2\hbar^2|\psi_{211}\rangle + 2\hbar^2|\psi_{210}\rangle - \sqrt{15}2\hbar^2|\psi_{21-1}\rangle]$$

$$= \frac{3 \times 3}{25} \times 2\hbar^2 + \frac{1}{25} \times 2\hbar^2 + \frac{15}{25} \times 2\hbar^2 = 2\hbar^2 \quad [\text{Using } \langle n\ell'm' | n\ell m \rangle = \delta_{\ell\ell'} \delta_{mm'}]$$

$$\langle L_z^2 \rangle = \langle \psi | L_z^2 | \psi \rangle$$

$$= \frac{1}{25}[\langle\psi_{211}|3 + \langle\psi_{210}| - \sqrt{15}\langle\psi_{21-1}|][3L_z^2|\psi_{211}\rangle + L_z^2|\psi_{210}\rangle - \sqrt{15}L_z^2|\psi_{21-1}\rangle]$$

$$= \frac{1}{25}[\langle\psi_{211}|3 + \langle\psi_{210}| - \sqrt{15}\langle\psi_{21-1}|][3\hbar^2\langle\psi_{211}| + 0 - \sqrt{15}\hbar^2\langle\psi_{21-1}|]$$

$$= \left(\frac{9}{25} + \frac{15}{25} \right) \hbar^2 = \frac{24}{25} \hbar^2$$

On substituting the values of $\langle L^2 \rangle$ and $\langle L_z^2 \rangle$ in equation (i), we get

$$\langle (L_x^2 + L_y^2) \rangle = 2\hbar^2 - \frac{24}{25} \hbar^2 = \frac{26}{25} \hbar^2$$

Correct answer is (d)

49. An oscillating current $I(t) = I_0 \exp(-i\omega t)$ flows in the direction of the y-axis through a thin metal sheet of area 1.0 cm^2 kept in the xy-plane. The rate of total energy radiated per unit area from the surfaces of the metal sheet at a distance of 100 m is

- (a) $I_0 \omega / (12\pi \epsilon_0 c^3)$ (b) $I_0^2 \omega^2 / (12\pi \epsilon_0 c^3)$
 (c) $I_0^2 \omega / (12\pi \epsilon_0 c^3)$ (d) $I_0 \omega^2 / (24\pi \epsilon_0 c^3)$

Soln. Current per unit length on sheet is $I(t) = I_0 e^{-i\omega t}$

Current per unit area of the sheet $\frac{I(t)}{\Delta y} = \frac{dp}{dt}$

Dipole moment per unit length per unit time $\dot{p} = \frac{dp}{dt} = I_0 e^{-i\omega t} \Rightarrow p = I_0 (-i\omega) e^{-i\omega t}$

Power radiated per unit area of the sheet is given by

$$P = \frac{\mu_0 \langle (\ddot{p})^2 \rangle}{6\pi c} = \frac{\mu_0 I_0^2 \langle e^{-2i\omega t} \rangle \omega^2}{6\pi c} \Rightarrow P = \frac{\mu_0 I_0^2 \omega^2}{12\pi c}$$

$$\Rightarrow P = \frac{\mu_0 \epsilon_0 I_0^2 \omega^2}{12\pi c \epsilon_0} = \frac{I_0^2 \omega^2}{12\pi \epsilon_0 c^3}$$

Correct answer is (b)

50. Consider a two-dimensional infinite square well $V(x, y) = \begin{cases} 0 & 0 < x < a, 0 < y < a \\ \infty & \text{otherwise} \end{cases}$. Its

normalized eigenfunctions are $\psi_{n_x, n_y}(x, y) = \frac{2}{a} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right)$ where $n_x, n_y = 1, 2, 3, \dots$. If a perturbation

$H' = \begin{cases} V_0 & 0 < x < \frac{a}{2}, 0 < y < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$ is applied, then the correction to the energy of the first excited

state to order V_0 is

- (a) $\frac{V_0}{4}$ (b) $\frac{V_0}{4} \left[1 \pm \frac{64}{9\pi^2} \right]$ (c) $\frac{V_0}{4} \left[1 \pm \frac{16}{9\pi^2} \right]$ (d) $\frac{V_0}{4} \left[1 \pm \frac{32}{9\pi^2} \right]$

Soln. The energy of n^{th} state for 2-dimensional infinite square well

$$E_n = \frac{n_x^2 \pi^2 \hbar^2}{2ma^2} + \frac{n_y^2 \pi^2 \hbar^2}{2ma^2} \quad \dots (i)$$

The energy of first excited state = $\frac{5\pi^2\hbar^2}{2ma^2}$

and state is 2 fold degenerate with $(n_x, n_y) = (1, 2) \& (2, 1)$

To find first order corection in every of first excited state we use degenerate perturbation theory.

We construct secular determinant

$$\begin{vmatrix} \langle \phi_1 | H_P | \phi_1 \rangle - E' & \langle \phi_1 | H_P | \phi_2 \rangle \\ \langle \phi_2 | H_P | \phi_1 \rangle & \langle \phi_2 | H_P | \phi_2 \rangle - E' \end{vmatrix} = 0 \quad \dots (ii)$$

Where, $\phi_1 = \frac{2}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)$ and $\phi_2 = \frac{2}{a} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)$

$$\langle \phi_1 | H_P | \phi_1 \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi_1^*(x, y) H_P \phi_1(x, y) dx dy$$

$$= V_0 \int_0^{a/2} \frac{2}{a^2} \sin^2\left(\frac{\pi x}{a}\right) dx \int_0^{a/2} \sin^2\left(\frac{2\pi y}{a}\right) dy = \frac{V_0}{4}$$

Similarly, $\langle \phi_2 | H_P | \phi_2 \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi_2^*(x, y) H_P \phi_2(x, y) dx dy = \frac{V_0}{4}$

$$\langle \phi_1 | H_P | \phi_2 \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi_1^*(x, y) H_P \phi_2(x, y) dx dy$$

$$= V_0 \int_0^{a/2} \int_0^{a/2} \left(\frac{2}{a}\right)^2 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) dx dy$$

$$= \frac{V_0}{a^2} \left[\int_0^{a/2} 2 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx \int_0^{a/2} 2 \sin\left(\frac{2\pi y}{a}\right) \sin\left(\frac{\pi y}{a}\right) dy \right]$$

$$= V_0 \frac{4}{3\pi} \times \frac{4}{3\pi} \frac{16V_0}{9\pi^2}$$

on substituting these values of matrix elements in secular determinant.

$$\begin{vmatrix} \frac{V_0}{4} - E' & \frac{16V_0}{9\pi^2} \\ \frac{16V_0}{9\pi^2} & \frac{V_0}{4} - E' \end{vmatrix} = 0 \Rightarrow \left(\frac{V_0}{4} - E'\right)^2 - \left(\frac{16V_0}{9\pi^2}\right)^2 = 0$$

$$\Rightarrow \frac{V_0}{4} - E' = \pm \frac{16V_0}{9\pi^2} \Rightarrow E' = \frac{V_0}{4} \pm \frac{16V_0}{9\pi^2} = \frac{V_0}{4} \left[1 \pm \frac{64}{9\pi^2} \right]$$

Correct answer is (b)

51. Consider a system of two Ising spins S_1 and S_2 taking value ± 1 with interaction energy given by $\varepsilon = -JS_1S_2$, when it is in thermal equilibrium at temperature T . For large T , the average energy of the system varies as $C/k_B T$, with C given by

- (1) $-2J^2$ (2) $-J^2$ (3) J^2 (4) $4J$

Soln. The interaction energy given by $\varepsilon = -JS_1S_2$

where S_1 and S_2 taking values ± 1 .

The possible values of the energy of the system are

$$E_1 = -J(+1)(+1) = -J \quad E_2 = -J(-1)(-1) = -J$$

$$E_3 = -J(+1)(-1) = +J \quad E_4 = -J(-1)(+1) = +J$$

When the system is in thermal equilibrium then the average energy of the system is given by

$$\langle E \rangle = \frac{\sum_r E_r g_r e^{-\beta E_r}}{\sum_r g_r e^{-\beta E_r}} = \frac{-2J e^{+\frac{J}{K_B T}} + 2J e^{-\frac{J}{K_B T}}}{2e^{+\frac{J}{K_B T}} + 2e^{-\frac{J}{K_B T}}} = \frac{-J[e^{\frac{J}{K_B T}} - e^{-\frac{J}{K_B T}}]}{[e^{\frac{J}{K_B T}} + e^{-\frac{J}{K_B T}}]}$$

For large T , $\frac{J}{K_B T} \ll 1$, $e^{\pm \frac{J}{K_B T}} = 1 \pm \frac{J}{K_B T}$ neglecting higher order terms

$$\langle E \rangle = \frac{-J \left[1 + \frac{J}{K_B T} - 1 + \frac{J}{K_B T} \right]}{1 + \frac{J}{K_B T} + 1 - \frac{J}{K_B T}} = \frac{-J^2}{K_B T} = \frac{C}{K_B T} = C = -J^2$$

Correct answer is (b)

52. Consider three particles A, B and C, each with an attribute S that can take two value ± 1 . Let $S_A = 1$, $S_B = 1$ and $S_C = -1$ at a given instant. In the next instant, each S value can change to $-S$ with probability $1/3$. The probability that $S_A + S_B + S_C$ remains unchanged is
 (a) $2/3$ (b) $1/3$ (c) $2/9$ (d) $4/9$

Soln. At a given instant, $S_A = 1$, $S_B = 1$, $S_C = -1 \Rightarrow S_A + S_B + S_C = 1$

At the next instant, $S_A + S_B + S_C$ will remain unchanged for the following cases:

S_A	S_B	S_C	$S_A + S_B + S_C$
1	1	-1	1
(unchanged)	(unchanged)	(unchanged)	
-1	1	-1	-1
(changed)	(unchanged)	(changed)	
1	-1	1	1
(unchanged)	(changed)	(changed)	

$$\text{Required probability} = P = \left(\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \right) + \left(\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} \right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \right) = \frac{4}{9}$$

Correct answer is (d)

53. The bound on the ground state energy of the Hamiltonian with an attractive delta-function potential, namely

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha \delta(x)$$

using the variational principle with the trial wavefunction $\psi(x) = A \exp(-bx^2)$ is

$$[\text{Note: } \int_0^\infty e^{-t} t^a dt = \Gamma(a+1)]$$

- (a) $-\frac{m\alpha^2}{4\pi\hbar^2}$ (b) $-\frac{m\alpha^2}{2\pi\hbar^2}$ (c) $-\frac{m\alpha^2}{\pi\hbar^2}$ (d) $-\frac{m\alpha^2}{\sqrt{5}\pi\hbar^2}$

Soln. The Hamiltonian, $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha\delta(x)$

The trial wave function $\psi(x) = A e^{-bx^2}$ with b variational parameter.

$$\begin{aligned} \langle H \rangle &= \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int_{-\infty}^{+\infty} A^* e^{-bx^2} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha\delta(x) \right) A e^{-bx^2} dx}{\int_{-\infty}^{+\infty} A^* e^{-bx^2} A e^{-bx^2} dx} \\ &= \frac{|A|^2 \int_{-\infty}^{+\infty} e^{-2bx^2} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha\delta(x) \right) dx}{|A|^2 \int_{-\infty}^{+\infty} e^{-2bx^2} dx} \\ &= \frac{-\frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} e^{-2bx^2} (4b^2 x^2 - 2b) e^{-2bx^2} dx - \alpha \int_{-\infty}^{+\infty} e^{-2bx^2} \delta(x) dx}{\int_{-\infty}^{+\infty} e^{-2bx^2} dx} \quad \left[\text{by using } \int_{-\infty}^{+\infty} f(x) \delta(x-a) dx = f(a) \right] \\ \langle \hat{H} \rangle &= \frac{-\frac{\hbar^2}{2m} 4b^2 \int_{-\infty}^{+\infty} e^{-2bx^2} x^2 dx + \frac{\hbar^2}{2m} 2b \int_{-\infty}^{+\infty} e^{-2bx^2} dx - \alpha \int_{-\infty}^{+\infty} e^{-2bx^2} \delta(x) dx}{\int_{-\infty}^{+\infty} e^{-2bx^2} dx} \\ &= \frac{-\frac{4b^2 \hbar^2}{2m} \times \frac{1}{2} \frac{\sqrt{\pi/2}}{(2b)^{3/2}} + \frac{2b \hbar^2}{2m} \times \frac{\sqrt{\pi}}{2(2b)^{1/2}} - \frac{\alpha}{2}}{\frac{\sqrt{\pi}}{2(2b)^{1/2}}} = -\frac{\hbar^2 b}{2m} + \frac{2b \hbar^2}{2m} - \frac{\alpha}{\sqrt{\pi}} (2b)^{1/2} = \frac{b \hbar^2}{2m} - \frac{\alpha (2b)^{1/2}}{\sqrt{\pi}} \end{aligned}$$

$$\begin{aligned} \frac{\partial \langle H \rangle}{\partial b} &= 0 \Rightarrow \frac{\hbar^2}{2m} - \frac{\alpha \times 2^{1/2}}{\sqrt{\pi}} \frac{1}{2b^{1/2}} = 0 \Rightarrow \frac{\alpha}{\sqrt{2\pi b}} = \frac{\hbar^2}{2m} \\ &\Rightarrow \frac{\alpha^2}{2\pi b} = \frac{\hbar^4}{4m^2} \Rightarrow b = \frac{2\alpha^2 m^2}{\pi \hbar^4} \end{aligned}$$

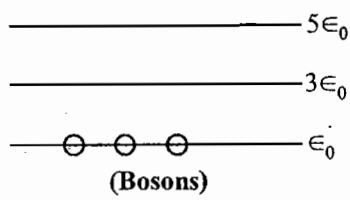
The ground state energy,

$$\begin{aligned} E &= \frac{\hbar^2}{2m} \left(\frac{2\alpha^2 m^2}{\pi \hbar^4} \right) - \frac{\alpha \times 2^{1/2}}{\sqrt{\pi}} \left(\frac{2\alpha^2 m^2}{\pi \hbar^4} \right)^{1/2} \\ &= \frac{\alpha^2 m}{\pi \hbar^2} - \frac{2\alpha^2 m}{\pi \hbar^2} = -\frac{\alpha^2 m}{\pi \hbar^2} \end{aligned}$$

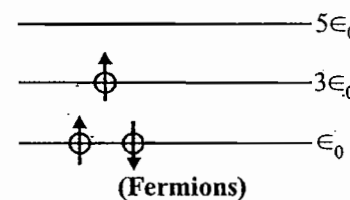
Correct answer is (c)

54. Consider two different systems each with three identical non-interacting particles. Both have single particle states with energies ϵ_0 , $3\epsilon_0$ and $5\epsilon_0$, ($\epsilon_0 > 0$). One system is populated by spin- $\frac{1}{2}$ fermions and the other by bosons. What is the value of $E_F - E_B$ where E_F and E_B are the ground state energies of the fermionic and bosonic systems respectively?
- (a) $6\epsilon_0$ (b) $2\epsilon_0$ (c) $4\epsilon_0$ (d) ϵ_0

Soln.



(Bosons)



(Fermions)

According to Pauli's exclusion principle

$$E_B = 3\epsilon_0$$

$$E_F = 2\epsilon_0 + 3\epsilon_0 + 5\epsilon_0$$

$$E_F - E_B = 5\epsilon_0 - 3\epsilon_0 = 2\epsilon_0$$

Correct answer is (b)

55. The input to a lock-in amplifier has the form $V_i(t) = V_i \sin(\omega t + \theta_i)$ where V_i , ω , θ_i are the amplitude, frequency and phase of the input signal respectively. This signal is multiplied by a reference signal of the same frequency ω , amplitude V_r and phase θ_r . If the multiplied signal is fed to a low pass filter of cut-off frequency ω , then the final output signal is

- (a) $\frac{1}{2} V_i V_r \cos(\theta_i - \theta_r)$ (b) $V_i V_r \left[\cos(\theta_i - \theta_r) - \cos\left(\frac{1}{2}\omega t + \theta_i + \theta_r\right) \right]$
- (c) $V_i V_r \sin(\theta_i - \theta_r)$ (d) $V_i V_r \left[\cos(\theta_i - \theta_r) + \cos\left(\frac{1}{2}\omega t + \theta_i + \theta_r\right) \right]$

Soln. After multiplication

$$V_0 = V_i \sin(\omega t + \theta_i) \cdot V_r \sin(\omega t + \theta_r)$$

$$\text{Resultant} = \frac{V_i V_r}{2} [\cos(\theta_i - \theta_r) - \cos(2\omega t + \theta_i + \theta_r)]$$

After low pass filter cut-off frequency ω 2nd term neglected.

$$V_0 = \frac{1}{2} V_i V_r [\cos(\theta_i - \theta_r)]$$

Correct answer is (a)

56. The solution of the partial differential equation

$$\frac{\partial^2}{\partial t^2} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = 0$$

satisfying the boundary conditions $u(0, t) = 0 = u(L, t)$ and initial conditions $u(x, 0) = \sin(\pi x/L)$

and $\frac{\partial}{\partial t} u(x, t)|_{t=0} = \sin(2\pi x/L)$ is

- (a) $\sin(\pi x/L) \cos(\pi t/L) + \frac{L}{2\pi} \sin(2\pi x/L) \cos(2\pi t/L)$

- (b) $2\sin(\pi x/L)\cos(\pi t/L) - \sin(\pi x/L)\cos(2\pi t/L)$
- (c) $\sin(\pi x/L)\cos(2\pi t/L) + \frac{L}{\pi}\sin(2\pi x/L)\sin(\pi t/L)$
- (d) $\sin(\pi x/L)\cos(\pi t/L) + \frac{L}{2\pi}\sin(2\pi x/L)\sin(2\pi t/L)$

Soln. The last initial condition is satisfied by only options (c) and (d).
Out of option (c) and (d), only option (d) satisfies the given partial differential equation.
Correct answer is (d)

57. Consider the hydrogen deuterium molecule HD. If the mean distance between the two atoms is 0.08 nm and the mass of the hydrogen atom is 938 MeV/c², then the energy difference ΔE between the two lowest rotational states is approximately
- (a) 10^{-1} eV (b) 10^{-2} eV (c) 2×10^{-2} eV (d) 10^{-3} eV

Soln. $r_e = 0.08 \text{ nm}$

the energy separation between two rotational states is given by

$$\Delta E = E_{J=1} - E_{J=0}$$

$$\text{But } E_J = \frac{h^2}{8\pi^2 I} J(J+1)$$

$$\Delta E = \frac{h^2}{8\pi^2 I} [2 - 0] = \frac{h^2}{4\pi^2 I}$$

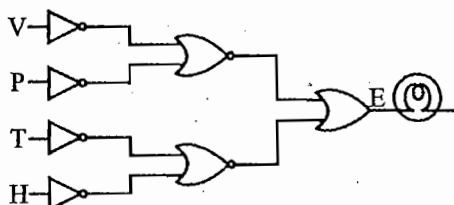
$$\text{Also } I = \mu r_e^2 \text{ where } \mu_{HD} = \frac{m_H m_D}{m_H + m_D} \approx \frac{2m_H m_H}{(m_H + 2m_H)} \approx \frac{2}{3} m_H$$

$$I = \frac{2}{3} \times 1.67 \times 10^{-27} \times 0.64 \times 10^{-20} \Rightarrow I = 7.117 \times 10^{-48} \text{ Kg.m}^2$$

$$\Delta E = \frac{(6.62)^2 \times 10^{-68}}{4 \times (3.14)^2 \times 7.12 \times 10^{-48}} = \frac{0.15606 \times 10^{-20} \text{ J}}{16 \times 10^{-19}} = 10^{-2} \text{ eV}$$

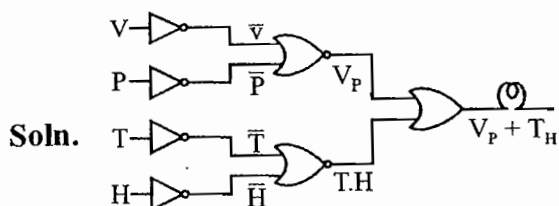
Correct answer is (b)

58. Four digital outputs V, P, T and H monitor the speed v, tyre pressure p, temperature t and relative humidity h of a car. These outputs switch from 0 to 1 when the values of the parameters exceed 85 km/hr, 2 bar, 40°C and 50%, respectively. A logic circuit that is used to switch ON a lamp at the output E is shown below.



Which of the following conditions will switch the lamp ON?

- (a) $v < 85 \text{ km/hr}, p < 2 \text{ bar}, t > 40^\circ \text{C}, h > 50\%$
- (b) $v < 85 \text{ km/hr}, p < 2 \text{ bar}, t > 40^\circ \text{C}, h < 50\%$
- (c) $v > 85 \text{ km/hr}, p < 2 \text{ bar}, t > 40^\circ \text{C}, h < 50\%$
- (d) $v > 85 \text{ km/hr}, p < 2 \text{ bar}, t < 40^\circ \text{C}, h > 50\%$



For the lamp to become on either $V_P = 1$ or $T.H = 1$ or both $= 1$

hence only option 1 have $t > 40^\circ$ and $n > 50\%$

hence 0 lamp is on only for option 1

Correct answer is (a)

59. The solution of the differential equation $\frac{dx}{dt} = x^2$ with the initial condition $x(0) = 1$ will blow up as t tends to
- (a) 1 (b) 2 (c) 1/2 (d) ∞

Soln. Given: $\frac{dx}{dt} = x^2 \Rightarrow \int \frac{dx}{x^2} = \int dt \Rightarrow -\frac{1}{x} = t + C$

Applying $x(0) = 1$, we get $C = 1 \Rightarrow -\frac{1}{x} = t + 1 \Rightarrow x = \frac{1}{1-t}$

Therefore, the solution of the given differential equation will blow up as t tends to 1.

Correct answer is (a)

60. Let u be a random variable uniformly distributed in the interval $[0, 1]$ and $V = -c \ln u$, where c is a real constant. If V is to be exponentially distributed in the interval $[0, \infty)$ with unit standard deviation, then the value of c should be
- (a) $\ln 2$ (b) 1/2 (c) 1 (d) -1

Soln. u be a random variable uniformly distributed in the interval $[0, 1]$ i.e. the distribution function will be given as

$$f(u) = \begin{cases} 1 & \text{for } [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

V is exponentially distributed in the interval $[0, \infty)$ with unit standard deviation i.e. the distribution function will be given as

$$f(V) = \lambda e^{-\lambda V} = e^{-V} \text{ for } [0, \infty)$$

Here, standard deviation of an exponential distribution is $\frac{1}{\lambda} = 1 \Rightarrow \lambda = 1$

To convert uniform distribution into exponential distribution, we have to assume

$$u = e^{-V} \Rightarrow V = -\ln u$$

Therefore, the value of c will be 1.

Correct answer is (c)

61. The inverse Laplace transform of $\frac{1}{s^2(s+1)}$ is

(a) $\frac{1}{2}t^2e^{-t}$ (b) $\frac{1}{2}t^2 + 1 - e^{-t}$ (c) $t - 1 + e^{-t}$ (d) $\frac{1}{2}t^2(1 - e^{-t})$

Soln. $f(s) = \frac{1}{s^2(s+1)} = \frac{s^2 - s(s+1) + (s+1)}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$

$$f(t) = L^{-1}f(s) = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right) = e^{-t} - 1 + t$$

Alternative way, Find the laplace tranform of the all options and it will be seen that laplace tranform of option (c) will be the one which is given in the question.

Correct answer is (c)

62. The number of degrees of freedom of a rigid body in d space-dimensions is
(a) 2d (b) 6 (c) d(d+1)/2 (d) d!

Soln. $DOF = \frac{d(d+1)}{2}$, d = dimension of space

In 1 - d, d = 1, $DOF = 1$

In 2 - d, d = 2, $DOF = 3$

In 3 - d, d = 3, $DOF = 6$

Correct answer is (c)

63. A particle of mass m is at the stable equilibrium position of its potential energy $V(x) = ax - bx^3$ where a, b are positive constants. The minimum velocity that has to be imparted to the particle to render its motion unstable is

(a) $(64a^3 / 9m^2b)^{1/4}$

(b) $(64a^3 / 27m^2b)^{1/4}$

(c) $(16a^3 / 27m^2b)^{1/4}$

(d) $(3a^3 / 64m^2b)^{1/4}$

Soln. $V(x) = ax - bx^3$
for stable position

$$\left. \frac{dV}{dx} \right|_{x=x_0} = 0 \Rightarrow a - 3bx_0^2 = 0$$

$$\Rightarrow x_0 = \pm \sqrt{\frac{a}{3b}} \Rightarrow \frac{d^2V}{dx^2} = -6bx$$

potential energy at stable position

(take $x_0 = -\sqrt{a/3b}$)

$$V(x_0) = ax_0 - bx_0^3 = x_0(a - bx_0^2) = -\sqrt{\frac{a}{3b}} \left(a - b \cdot \frac{a}{3b} \right) = -\sqrt{\frac{a}{3b}} \cdot \frac{2a}{3}$$

To make motion unstable particle should at least reach unstable point.

Therefore, loss in K.E. = gain in P.E. $\frac{1}{2}mv^2 = \frac{4a}{3} \sqrt{\frac{a}{3b}}$

This gives, $v = \sqrt{\frac{64a^3}{27bm^2}}$

Correct answer is (b)

64. If the operators A and B satisfy the commutation relation $[A, B] = I$, where I is the identity operator, then

(a) $[e^A, B] = e^A$ (b) $[e^A, B] = [e^B, A]$
 (c) $[e^A, B] = [e^{-B}, A]$ (d) $[e^A, B] = 1$

Soln. $[A, B] = I$ & $e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$

$$[e^A, B] = \left[1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots, B \right]$$

$$= [1, B] + [A, B] + \frac{1}{2!} [A^2, B] + \frac{1}{3!} [A^3, B] + \dots$$

$$[1, B] = B - B = 0$$

$$[A^2, B] = 2[A, B]A = 2A$$

$$[A^3, B] = 3A^2[A, B] = 3A^2$$

⋮

$$[e^A, B] = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots = e^A$$

Correct answer is (a)

65. A system is governed by the Hamiltonian $H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2$ where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved?

(a) $a = -3, b = 2$ (b) $a = 3, b = -2$ (c) $a = 2, b = -3$ (d) $a = -2, b = 3$

Soln. $P_x - 3y$ and $P_y + 2x$ are constants of motion

$$\therefore \{p_x - 3y, H\} = 0$$

$$\text{Let } p_x - 3y = \alpha$$

$$\therefore \{\alpha, H\} = 0$$

$$\therefore \frac{\partial \alpha}{\partial x} \frac{\partial H}{\partial p_x} - \frac{\partial \alpha}{\partial p_x} \frac{\partial H}{\partial x} + \frac{\partial \alpha}{\partial y} \frac{\partial H}{\partial p_y} - \frac{\partial \alpha}{\partial p_y} \frac{\partial H}{\partial y} = 0$$

$$\Rightarrow 0 + b(p_y - bx) + (-3)(p_y - bx) - 0$$

$$\Rightarrow (p_y - bx)(b - 3) = 0$$

$$p_y - bx \neq 0$$

$$\therefore b = 3$$

$$\text{Similarly using } \{p_y + 2x, H\} = 0$$

We get value of a .

Correct answer is (d)

66. Using the frequency-dependent Drude formula, what is the effective kinetic inductance of a metallic wire that is to be used as a transmission line? [In the following, the electron mass is m , density of electrons is n , and the length and cross-sectional area of the wire are ℓ and A respectively]

(a) $mA/(ne^2\ell)$

(b) zero

(c) $m\ell/(ne^2A)$

(d) $m\sqrt{A}/(ne^2\ell^2)$

Soln. According to energy conservation $N \frac{1}{2} m v_d^2 = \frac{1}{2} L I^2$

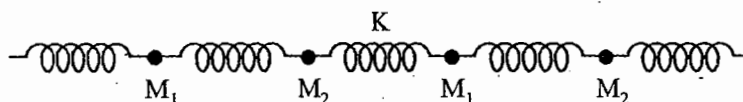
Where $N \rightarrow$ total number of electrons, $v_d \rightarrow$ drift velocity of electron, $L \rightarrow$ inductance

$$n A l m v_d^2 = L n^2 e^2 A^2 v_d^2$$

$$\Rightarrow L = \frac{m l}{n e^2 A^2}$$

Correct answer is (c)

67. The phonon dispersion for the following one-dimensional diatomic lattice with masses M_1 and M_2 (as shown in the figure)



is given by

$$\omega^2(q) = K \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \left[1 \pm \sqrt{1 - \frac{4M_1M_2}{(M_1+M_2)^2} \sin^2\left(\frac{qa}{2}\right)} \right]$$

where a is the lattice parameter and K is spring constant. The velocity of sound is

(a) $\sqrt{\frac{K(M_1+M_2)}{2M_1M_2}} a$

(c) $\sqrt{\frac{K(M_1+M_2)}{M_1M_2}} a$

(b) $\sqrt{\frac{K}{2(M_1+M_2)}} a$

(d) $\sqrt{\frac{KM_1M_2}{2(M_1+M_2)^3}} a$

Soln. $\omega^2(q) = K \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \left[1 \pm \sqrt{1 - \frac{4M_1M_2}{(M_1+M_2)^2} \sin^2\left(\frac{qa}{2}\right)} \right]$

for $|qa| \ll \pi$, $\sin\left(\frac{qa}{2}\right) \rightarrow \left(\frac{qa}{2}\right)$

Take $-$ sign for acoustical branch

$$\omega^2(q) = K \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \left[1 - \left(1 - \frac{4M_1M_2}{(M_1+M_2)^2} \frac{q^2 a^2}{4} \right)^{1/2} \right]$$

$$\omega^2(q) = K \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \left[1 - 1 + \frac{1}{2} \frac{M_1M_2 q^2 a^2}{(M_1+M_2)^2} \right]$$

$$= K \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \left[\frac{1}{2} \frac{q^2 a^2}{M_1M_2 \left(\frac{1}{M_1} + \frac{1}{M_2} \right)^2} \right] = \frac{1}{2} K \frac{q^2 a^2}{M_1M_2 \left(\frac{M_1+M_2}{M_1M_2} \right)}$$

$$\Rightarrow \omega^2(q) = \frac{1}{2} K \frac{q^2 a^2}{(M_1 + M_2)}$$

$$\Rightarrow \omega(q) = \pm \left(\frac{1}{2} \frac{K a^2}{(M_1 + M_2)} \right)^{1/2} q$$

$$\text{Therefore, } v_{\text{sound}} = \frac{\omega}{q} = \sqrt{\frac{K}{2(M_1 + M_2)}} a$$

Correct answer is (b)

68. The binding energy of a light nucleus (Z, A) in MeV is given by the approximate formula

$$B(A, Z) \approx 16A - 20A^{2/3} - \frac{3}{4} Z^2 A^{-1/3} + 30 \frac{(N - Z)^2}{A}$$

where $N = A - Z$ is the neutron number. The value of Z of the most stable isobar for a given A is

- (a) $\frac{A}{2} \left(1 - \frac{A^{2/3}}{160} \right)^{-1}$ (b) $\frac{A}{2}$ (c) $\frac{A}{2} \left(1 - \frac{A^{2/3}}{120} \right)^{-1}$ (d) $\frac{A}{2} \left(1 + \frac{A^{4/3}}{64} \right)$

Soln. $B(A, Z) = 16A - 20A^{2/3} - \frac{3}{4} Z^2 A^{-1/3} + 30 \frac{(A - 2Z)^2}{A}$

For most stable isobar $\frac{dB}{dZ} = 0$

$$\Rightarrow -\frac{3}{4} A^{-1/3} (2Z) + \left(\frac{30}{A} \right) 2(A - 2Z)(-2) = 0$$

$$\Rightarrow -A^{-1/3} Z = \frac{80}{A} (A - 2Z)$$

$$\Rightarrow Z = \frac{80A}{160 - A^{2/3}} = \frac{80A}{160 \left[1 - \frac{A^{2/3}}{160} \right]} = \frac{A}{2} \left[1 - \frac{A^{2/3}}{160} \right]^{-1}$$

(Note: In nuclear physics, the sign of the last term should be negative in that case the answer will

$$\text{be } \frac{A}{2} \left[1 + \frac{A^{2/3}}{160} \right]^{-1})$$

Correct answer is (a)

69. Muons are produced through the annihilation of particle a and its antiparticle, namely the process $a + \bar{a} \rightarrow \mu^+ + \mu^-$. A muon has a rest mass of $105 \text{ MeV}/c^2$ and its proper life time is $2 \mu\text{s}$. If the center of mass energy of the collision is 2.1 GeV in the laboratory frame that coincides with the center-of-mass frame, then the fraction of muons that will decay before they reach a detector placed 6 km away from the interaction point is

- (a) e^{-1} (b) $1 - e^{-1}$ (c) $1 - e^{-2}$ (d) e^{-10}

Soln. $N = N_0 e^{-\lambda t}$

$$\frac{N}{N_0} = e^{-\lambda t} = e^{-\frac{t}{\tau}}, \text{ where } \tau \text{ is proper time} = 2 \times 10^{-6} \text{ s.}$$

Dilated time = $\gamma \tau$, where $\gamma = \frac{(2.1) \times 10^3}{105} = 20$

Now, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, since $\gamma = 20$ then $\beta \approx 1$ or $v = C$

Now, $\frac{N}{N_0} = \exp\left(-\frac{6 \times 10^3}{3 \times 10^8 \times 2.0 \times 10^{-6} \times 20}\right) = \exp(-0.5) = 0.606 \approx 1 - e^{-1}$

Correct answer is (b)

(Note: If the correct proper life time = $2.2 \mu\text{s}$ is given, then the answer will be exact)

70. The conductors in a 0.75 km long two-wire transmission line are separated by a centre-to-centre distance of 0.2m. If each conductor has a diameter of 4cm, then the capacitance of the line is
 (a) $8.85 \mu\text{F}$ (b) 88.5 nF (c) 8.85 pF (d) 8.85 nF

Soln. Capacitance per unit length of two wire line is given by

$$C' = \frac{\pi \epsilon}{\ln\left(\frac{2D}{d}\right)} = \frac{3.14 \times 8.85 \times 10^{-12}}{\ln\left(\frac{2 \times 0.2}{4 \times 10^{-2}}\right)} \text{ Farad/m}$$

$$\Rightarrow C' = \frac{3.14 \times 8.85 \times 10^{-12}}{\ln(10)} = 106.8071 \times 10^{-12} \text{ Farad/m}$$

Total capacitance $C = C' \times 0.75 \times 10^3 \text{ Farad}$

$C = 80.11 \times 10^{-9} \text{ Farad} = 80.11 \text{ nF}$

Correct answer is (b)

71. The electron dispersion relation for a one-dimensional metal is given by

$$\epsilon_k = 2\epsilon_0 \left[\sin^2 \frac{ka}{2} - \frac{1}{6} \sin^2 ka \right]$$

where k is the momentum, a is the lattice constant, ϵ_0 is a constant having dimensions of energy and $|ka| \leq \pi$. If the average number of electrons per atom in the conduction band is $1/3$, then the Fermi energy is

- (a) $\epsilon_0 / 4$ (b) ϵ_0 (c) $2\epsilon_0 / 3$ (d) $5\epsilon_0 / 3$

72. The electronic energy levels in a hydrogen atom are given by $E_n = -13.6/n^2 \text{ eV}$. If a selective excitation to the $n = 100$ level is to be made using a laser, the maximum allowed frequency line-width of the laser is

- (a) 6.5 MHz (b) 6.5 GHz (c) 6.5 Hz (d) 6.5 kHz

Soln. $E_n = -\frac{13.6}{n^2} \text{ eV} \Rightarrow \Delta E_n = +\frac{13.6 \times 2}{n^3} \text{ eV}$

$$\Rightarrow h\Delta\nu = \frac{13.6 \times 2}{n^3} \times 1.6 \times 10^{-19}$$

Frequency line-width

$$\Rightarrow \Delta\nu = \frac{13.6 \times 2 \times 16 \times 10^{-19}}{10^6 \times 6.6 \times 10^{-34}} \text{ Hz} \Rightarrow \Delta\nu = 6.59 \times 10^9 \text{ Hz} \Rightarrow \Delta\nu = 6.59 \text{ GHz}$$

Correct answer is (b)

73. If the energy dispersion of a two-dimensional electron system is $E = u\hbar k$ where u is the velocity and k is the momentum, then the density of states $D(E)$ depends on the energy as
 (a) $1/\sqrt{E}$ (b) $\sqrt{E}$ (c) E (d) constant

Soln. If $E = u\hbar k$, where u is the velocity and k is momentum, k is the momentum,

The volume of phase space in 2-dimension

= position space volume $\times$ momentum space volume

= $A \times \pi p^2$ [for sharp value of momentum]

The number of microstates

$$\Omega = \frac{A\pi p^2}{n^2} = \frac{A\pi(\hbar k^2)}{h^2} \text{ (for sharp value of } k\text{)}$$

$$= \frac{A\pi E^2}{h^2 u^2} \text{ (for sharp value of energy)}$$

If energy lies between E to $E + dE$, then number of microstates,

$$\Omega' = \frac{2A\pi E}{h^2 u^2} dE = \partial(E) dE$$

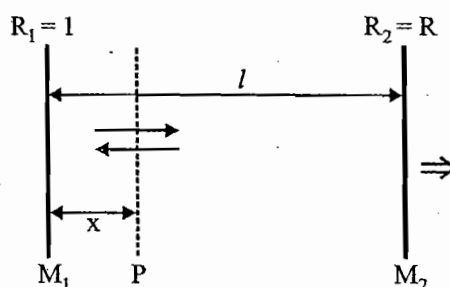
The density of states = Number of microstates or energy states per unit energy interval.

$$D(E) = \frac{2A\pi E}{h^2 u^2}$$

$$D(E) \propto E$$

Correct answer is (c)

74. Consider the laser resonator cavity shown in the figure. If I_1 is the intensity of the radiation at mirror M_1 and α is the gain coefficient of the medium between the mirrors, then the energy density of photons in the plane P at a distance x from M_1 is



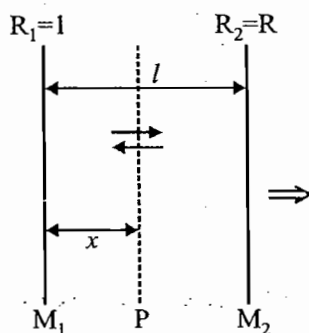
(a) $(I_1/c)e^{-\alpha x}$

(b) $(I_1/c)e^{\alpha x}$

(c) $(I_1/c)(e^{\alpha x} + e^{-\alpha x})$

(d) $(I_1/c)e^{2\alpha x}$

Soln. Energy density of the standing wave between the mirrors is given by



$$\frac{I_1}{C}(e^{ax} + e^{-ax})$$

Correct answer is (c)

75. A spin-1/2 particle A undergoes the decay $A \rightarrow B + C + D$ where it is known that B and C are also spin-1/2 particles. The complete set of allowed values of the spin of the particle D is

- (a) $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \dots$ (b) 0, 1 (c) $\frac{1}{2}$ only (d) $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$

Soln. $A \rightarrow B + C + D$

It is like the decay of a neutron

$n \rightarrow p + e^- + \bar{\nu}_e$, where spin of each particle = 1/2.

Spin of the left side and the combined spin of the products must be same to conserve the spin angular momentum conservation law.

$$B \otimes C = \frac{1}{2} \otimes \frac{1}{2} = 0, 1 \text{ (Singlet, triplet)}$$

$$(B \otimes C) \otimes D = 0 \otimes \frac{1}{2} = \frac{1}{2} \text{ (Doublet)}$$

$$\text{or } 1 \otimes \frac{1}{2} = \frac{1}{2}, \frac{3}{2} \text{ (Doublet, Quartet)}$$

So from B, C, D, we can form either two doublets or a quartet spin state

Since A has spin = 1/2, $\therefore$ spin of D = $\frac{1}{2}$ only

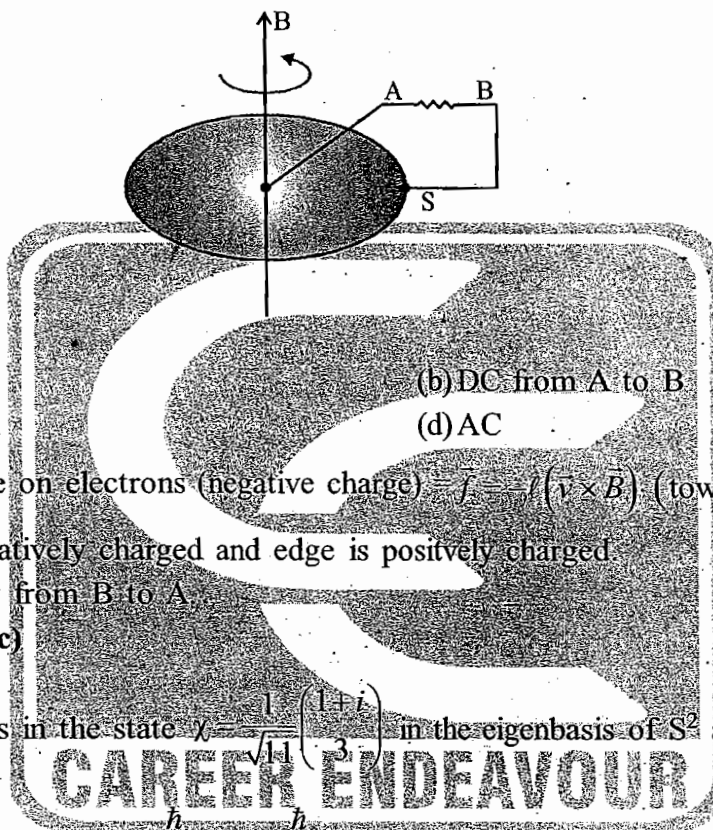
Correct answer is (c)

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PHYSICAL SCIENCES BOOKLET - [C]

Part-B

21. A horizontal metal disc rotates about the vertical axis in a uniform magnetic field pointing up as shown in the figure. A circuit is made by connecting one end A of a resistor to the centre of the disc and the other end B to its edge through a sliding contact S. The current that flows through the resistor is



- (a) zero
 (b) DC from A to B
 (c) DC from B to A
 (d) AC

Soln. Due to rotation force on electrons (negative charge) $= \vec{f} = -e(\vec{v} \times \vec{B})$ (towards axis of rotation)

Hence, centre is negatively charged and edge is positively charged.

So, current will flow from B to A.

Correct answer is (c)

22. A spin $-\frac{1}{2}$ particle is in the state $\chi = \frac{1}{\sqrt{11}} \begin{pmatrix} 1+i \\ 3 \end{pmatrix}$ in the eigenbasis of S^2 and S_z . If we measure S_z the probabilities of getting $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, respectively, are

- (a) $\frac{1}{2}$ and $\frac{1}{2}$ (b) $\frac{2}{11}$ and $\frac{9}{11}$ (c) 0 and 1 (d) $\frac{1}{11}$ and $\frac{3}{11}$

Soln. Given : A spin $\frac{1}{2}$ particle is in the state $\chi = \frac{1}{\sqrt{11}} \begin{pmatrix} 1+i \\ 3 \end{pmatrix}$

Measurement of S_z gives $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ with respective eigenbasis $\chi_{1/2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\chi_{-1/2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Method-1:

Representing $\frac{1}{\sqrt{11}} \begin{pmatrix} 1+i \\ 3 \end{pmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow C_1 = \frac{1+i}{\sqrt{11}}, C_2 = \frac{3}{\sqrt{11}}$

Probability of finding S_z as $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are $|C_1|^2 = \frac{2}{11}$ and $|C_2|^2 = \frac{9}{11}$ respectively.

Method-2:

$$\text{Initial state of particle} = |\psi_i\rangle = \frac{1}{\sqrt{11}} \begin{pmatrix} 1+i \\ 3 \end{pmatrix}$$

$$\text{Final state of particle} = |\psi_{f_1}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |\psi_{f_2}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Probability of finding } S_z \text{ as } +\frac{\hbar}{2} \text{ and } -\frac{\hbar}{2} \text{ are } \left| \langle \psi_{f_1} | \psi_i \rangle \right|^2 = \frac{2}{11} \text{ and } \left| \langle \psi_{f_2} | \psi_i \rangle \right|^2 = \frac{9}{11} \text{ respectively.}$$

Correct answer is (b)

23. Which of the following functions cannot be the real part of a complex analytic function of $z = x + iy$?

(a) x^2y (b) $x^2 - y^2$ (c) $x^3 - 3xy^2$ (d) $3x^2y - y - y^3$

Soln. The real part and imaginary part of a complex analytic function of $z = x + iy$, will be harmonic in nature i.e. both will satisfy Laplace's equation. Suppose, 'u' be the real part, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\text{Since, } \frac{\partial^2 (x^2y)}{\partial x^2} + \frac{\partial^2 (x^2y)}{\partial y^2} = 2y \neq 0$$

Correct answer is (a)

24. The motion of a particle of mass m in one dimension is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x. \text{ What is the difference between the (quantized) energies of the first two levels? (In the following, } \langle x \rangle \text{ is the expectation value of } x \text{ in the ground state)}$$

(a) $\hbar\omega - \lambda\langle x \rangle$ (b) $\hbar\omega + \lambda\langle x \rangle$ (c) $\hbar\omega + \frac{\lambda^2}{2m\omega^2}$ (d) $\hbar\omega$

Soln. Given : Hamiltonian of the system : $H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}_{\text{Unperturbed Hamiltonian}} + \underbrace{\lambda x}_{\text{Perturbed Hamiltonian}}$

First order correction to energy of the n^{th} state.

$$E_n^{(1)} = \langle n | \lambda x | n \rangle = \lambda \sqrt{\frac{\hbar}{2m\omega}} \langle n | (\hat{a} + \hat{a}^\dagger) | n \rangle = 0$$

Second order correction to energy of the n^{th} state.

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n | \lambda x | m \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \lambda^2 \frac{\hbar}{2m\omega} \sum_{m \neq n} \frac{|\langle n | (\hat{a} + \hat{a}^\dagger) | m \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

The non-zero contribution comes only from $m = (n+1)$ and $(n-1)$

$$\Rightarrow E_n^{(2)} = \frac{\lambda^2 \hbar}{2m\omega} \left[\frac{(\sqrt{n+1})^2}{-\hbar\omega} + \frac{(\sqrt{n})^2}{\hbar\omega} \right] = -\frac{\lambda^2}{2m\omega^2}$$

$$\text{So, } E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} = \left(n + \frac{1}{2} \right) \hbar\omega - \frac{\lambda^2}{2m\omega^2}$$

Energy difference between two levels is

$$(E_1 - E_0) = \left(\frac{3}{2} \hbar\omega - \frac{\lambda^2}{2m\omega^2} \right) - \left(\frac{1}{2} \hbar\omega - \frac{\lambda^2}{2m\omega^2} \right) = \hbar\omega$$

Correct answer is (d)

25. Let ψ_{nlm} denote the eigenfunctions of a Hamiltonian for a spherically symmetric potential $V(r)$. The expectation value of L_z in the state

$$\psi = \frac{1}{6} [\psi_{200} + \sqrt{5} \psi_{210} + \sqrt{10} \psi_{21-1} + \sqrt{20} \psi_{211}] \text{ is}$$

- (a) $-\frac{5}{18} \hbar$ (b) $\frac{5}{6} \hbar$ (c) $\hbar$ (d) $\frac{5}{18} \hbar$

Soln. Given: $\psi = \frac{1}{6} [\psi_{200} + \sqrt{5} \psi_{210} + \sqrt{10} \psi_{21-1} + \sqrt{20} \psi_{211}]$

$$\text{Since, } \hat{L}_z \psi_{nlm} = m \hbar \psi_{nlm}$$

$$\text{Therefore, } \langle L_z \rangle = \sum_i P_i (L_z)_i = \frac{1}{36} (0) + \frac{5}{36} (0) + \frac{10}{36} (-\hbar) + \frac{20}{36} (\hbar)$$

$$\Rightarrow \langle L_z \rangle = \frac{5}{18} \hbar$$

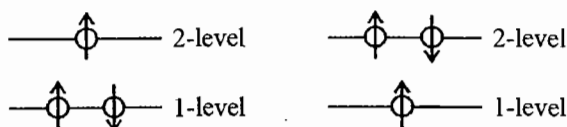
Correct answer is (d)

26. Three identical spin $-\frac{1}{2}$ fermions are to be distributed in two non-degenerate distinct energy levels. The number of ways this can be done is

- (a) 8 (b) 4 (c) 3 (d) 2

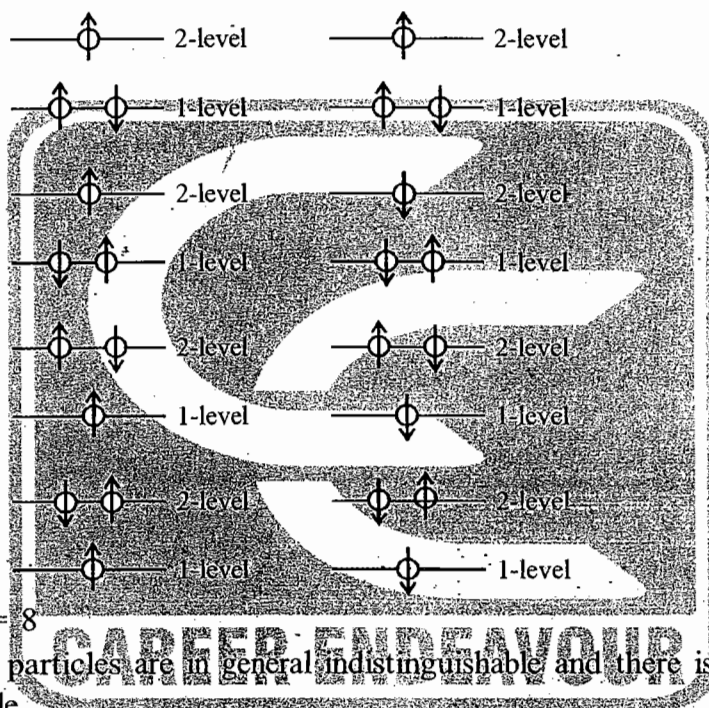
Soln. The maximum number of particles with spin's can be filled in a non-degenerate energy level = $2s + 1$.

Fermions are quantum particles and quantum identical particles are in general indistinguishable. If we include the concept of indistinguishability then three identical spin- $\frac{1}{2}$ particles (fermions) can be distributed in two non-degenerate distinct energy levels as



The number of ways = 2

On changing the direction of spin or orientation of spin, all the distribution will be identical. If we include the concept of distinguishability due to spin or we distinguish the particles due to spin the 3-identical particles with spin- $\frac{1}{2}$ can be distributed in two non-degenerate energy levels as



The number of ways = 8

But quantum identical particles are in general indistinguishable and there is no hint to consider particles distinguishable.

Correct answer is (d)

27. Let A , B and C be functions of phase space variables (coordinates and momenta of a mechanical system). If $\{, \}$ represents the Poisson bracket, the value of $\{A, \{B, C\}\} - \{\{A, B\}, C\}$ is given by
- (a) 0 (b) $\{B, \{C, A\}\}$ (c) $\{A, \{C, B\}\}$ (d) $\{\{C, A\}, B\}$

Soln. $y = \{A, \{B, C\}\} - \{\{A, B\}, C\}$

$\therefore y = \{A, \{B, C\}\} + \{C, \{A, B\}\}$

Jacobi identity

$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$

$\therefore \{A, \{B, C\}\} + \{C, \{A, B\}\} = -\{B, \{C, A\}\}$

$\therefore y = -\{B, \{C, A\}\} = \{\{C, A\}, B\}$

Correct answer is (d)

28. If A, B and C are non-zero Hermitian operators, which of the following relations must be false?

- (a) $[A, B] = C$ (b) $AB + BA = C$ (c) $ABA = C$ (d) $A + B = C$

Soln. Given : $A^\dagger = A, B^\dagger = B, C^\dagger = C$

$$C = [A, B] = AB - BA \Rightarrow C^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB = -C$$

So, $C = [A, B]$ is not hermitian.

Correct answer is (a)

29. The expression

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} \right) \frac{1}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)} \text{ is proportional to}$$

- (a) $\delta(x_1 + x_2 + x_3 + x_4)$ (b) $\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$
 (c) $(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{3/2}$ (d) $(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{-3/2}$

Soln.

$$\begin{aligned} & \frac{\partial^2}{\partial x_1^2} \left[\frac{1}{x_1^2 + x_2^2 + x_3^2 + x_4^2} \right] = \frac{\partial}{\partial x_1} \left[-\frac{2x_1}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2} \right] \\ & = -2 \left[\frac{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 - 4x_1^2(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^4} \right] \\ & = \frac{8x_1^2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - 2(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3} = \frac{6x_1^2 - 2x_2^2 - 2x_3^2 - 2x_4^2}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3} \end{aligned}$$

$$\text{Similarly, } \frac{\partial^2}{\partial x_2^2} \left[\frac{1}{x_1^2 + x_3^2 + x_2^2 + x_4^2} \right] = \frac{6x_2^2 - 2x_1^2 - 2x_3^2 - 2x_4^2}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^3}$$

and for others we can also calculate.

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} \right) \left(\frac{1}{x_1^2 + x_2^2 + x_3^2 + x_4^2} \right) = 0 \text{ for } x_1, x_2, x_3, x_4 \neq 0$$

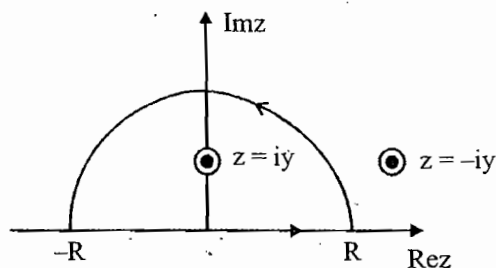
Correct answer is (b)

30. Given that the integral $\int_0^\infty \frac{dx}{y^2 + x^2} = \frac{\pi}{2y}$, the value of $\int_0^\infty \frac{dx}{(y^2 + x^2)^2}$ is

- (a) $\frac{\pi}{y^3}$ (b) $\frac{\pi}{4y^3}$ (c) $\frac{\pi}{8y^3}$ (d) $\frac{\pi}{2y^3}$

Soln.
$$\int_0^{\infty} \frac{dx}{(y^2 + x^2)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(y^2 + x^2)^2}$$

So, $f(z) = \frac{1}{(z^2 + y^2)^2}$ has poles of order '2' at $z \pm iy$



Since, both poles are not on real axis, then given integral

$$I = \frac{1}{2} [2\pi i (\text{Residue of } f(z) \text{ at } z = iy)]$$

$$= \frac{\pi i}{(2-1)!} \left[\frac{d}{dz} \left\{ (z - iy)^2 \frac{1}{(z^2 + y^2)^2} \right\} \right]_{z=iy} = \pi i \left[\frac{2}{(z + iy)^3} \right]_{z=iy} = \frac{\pi}{4y^3}$$

Correct answer is (b)

31. The force between two long and parallel wires carrying currents I_1 and I_2 and separated by a distance D is proportional to
- (a) $I_1 I_2 / D$ (b) $(I_1 + I_2) / D$ (c) $(I_1 I_2 / D)^2$ (d) $I_1 I_2 / D^2$

Soln. Force per unit length on each wire is given by

$$f = \frac{\mu_0 I_1 I_2}{4\pi D}$$



Correct answer is (a)

32. A loaded dice has the probabilities $\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}$ and $\frac{6}{21}$ of turning up 1, 2, 3, 4, 5 and 6, respectively. If it is thrown twice, what is the probability that the sum of the numbers that turn up is even?

- (a) $\frac{144}{441}$ (b) $\frac{225}{441}$ (c) $\frac{221}{441}$ (d) $\frac{220}{441}$

Soln. If the loaded dice is thrown twice, the sum of the numbers that turn up will be even if both number will be even or both will be odd.

Required probability = $P(\text{Even} - \text{Even}) + P(\text{Odd} - \text{Odd})$

$$= \left(\frac{12}{21} \times \frac{12}{21} \right) + \left(\frac{9}{21} \times \frac{9}{21} \right) = \frac{225}{441}$$

Correct answer is (b)

33. A particle moves in a potential $V = x^2 + y^2 + \frac{z^2}{2}$. Which component (s) of the angular momentum is / are constant (s) of motion?
 (a) none (b) L_x , L_y and L_z (c) only L_x and L_y (d) only L_z

Soln. Given $V(x, y, z) = x^2 + y^2 + \frac{z^2}{2}$

In polar co-ordinate (spherical polar)

$$V = r^2 \sin^2 \theta + \frac{r^2 \cos^2 \theta}{2} \quad \begin{cases} x = r \cos \phi \sin \theta \\ y = r \sin \phi \sin \theta \\ z = r \cos \theta \end{cases}$$

Therefore, Lagrangian of system is

$$L = T - V$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - r^2 \sin^2 \theta - \frac{r^2 \cos^2 \theta}{2}$$

ϕ is cyclic, therefore p_ϕ is constant of motion.

p_ϕ is equal to L_z . Therefore, L_z is constant of motion.

Correct answer is (d)

34. The Hamiltonian of a relativistic particle of rest mass m and momentum p is given by $H = \sqrt{p^2 + m^2} + V(x)$, in units in which the speed of light $c = 1$. The corresponding Lagrangian is

- (a) $L = m\sqrt{1 + \dot{x}^2} - V(x)$ (b) $L = -m\sqrt{1 - \dot{x}^2} - V(x)$
 (c) $L = \sqrt{1 + m\dot{x}^2} - V(x)$ (d) $L = \frac{1}{2} m\dot{x}^2 - V(x)$

Soln. $H = \sqrt{p^2 + m^2} + V(x)$

This is Hamiltonian of a relativistic particle with $c = 1$. It is very well known that Lagrangian of a relativistic particle is

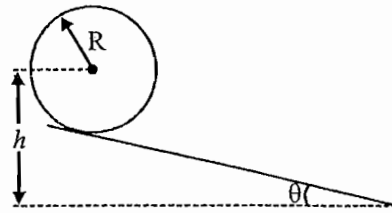
$$L = -mc^2 \sqrt{1 - v^2/c^2} - V(x)$$

Taking $v = \dot{x}$, and $c = 1$, we get

$$L = -m\sqrt{1 - \dot{x}^2} - V(x)$$

Correct answer is (b)

35. A ring of mass m and radius R rolls (without slipping) down an inclined plane starting from rest. If the centre of the ring is initially at a height h , the angular velocity when the ring reaches the base is



(a) $\sqrt{g/(h-R)} \tan \theta$

(b) $\sqrt{g/(h-R)}$

(c) $\sqrt{g(h-R)/R^2}$

(d) $\sqrt{2g/(h-R)}$

Soln. Let v = speed of the ring when it reaches the base.

Therefore, total K.E. of the ring $= \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2 = \frac{1}{2} m R^2 \cdot \frac{v^2}{R^2} + \frac{1}{2} m v^2 = m v^2$

Height descended by the centre of the ring = $h-R$

Therefore, loss in P.E. = gain in K.E.

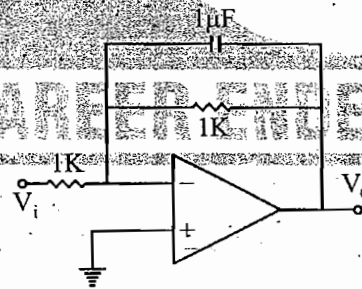
$$m g (h-R) = m v^2$$

$$v = \sqrt{g(h-R)}$$

Angular velocity $\omega = \frac{v}{R} = \sqrt{\frac{g(h-R)}{R^2}}$

Correct answer is (c)

36. Consider the op-amp circuit shown in the figure.



If the input is a sinusoidal wave $V_i = 5 \sin(1000t)$, then the amplitude of the output V_o is

(a) $\frac{5}{2}$

(b) 5

(c) $\frac{5\sqrt{2}}{2}$

(d) $5\sqrt{2}$

Soln. $X_C = \frac{1}{j\omega C} = \frac{1}{j \times 10^3 \times 10^{-6}} = -j k\Omega$

$$Z_F = X_C \parallel 1k = \frac{-j \times 1}{1-j} = \frac{-j}{1-j}$$

$$\frac{V_o}{V_i} = \frac{-Z_F}{R} = \frac{-\left(\frac{-j}{1-j}\right)}{1}$$

$$\frac{V_0}{V_i} = \frac{J}{1-J} \Rightarrow \left| \frac{V_0}{V_i} \right| = \frac{1}{\sqrt{1+1}} \Rightarrow \left| \frac{V_0}{V_i} \right| = \frac{1}{\sqrt{2}}$$

$$\therefore \begin{cases} V_i = 5 \sin(100t), & \omega = 10^3 \\ |V_i| = 5 \end{cases}$$

$$\therefore V_0 = \frac{1}{\sqrt{2}} \times |V_i| \Rightarrow V_0 = \frac{5}{\sqrt{2}} \Rightarrow V_0 = \frac{5 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \Rightarrow \boxed{V_0 = \frac{5\sqrt{2}}{2}}$$

Correct answer is (c)

37. If one of the inputs of a J-K flip flop is high and the other is low, then the outputs Q and $\bar{Q}$
- oscillate between low and high in race-around condition
 - toggle and the circuit acts like a T flip flop
 - are opposite to the inputs
 - follow the inputs and the circuit acts like an R-S flip flop

Soln. J-K flip-flop follow the I/P and the circuit acts like an R-S flip flop

Through table of JK =

J	K	Q_{n+1}
0	0	Q_n
0	1	0
1	0	1
1	1	Q_n

Truth table of RS =

S	R	Q_{n+1}
0	0	Q_n
0	1	0
1	0	1
1	1	Invalid

Correct answer is (d)

38. Two monochromatic sources, L_1 , and L_2 , emit light at 600 and 700 nm, respectively. If their frequency bandwidths are 10^{-1} and 10^{-3} GHz, respectively, then the ratio of linewidth of L_1 and L_2 is approximately
- 100 : 1
 - 1 : 85
 - 75 : 1
 - 1 : 75

Soln. $\lambda_1 = 600 \text{ nm}$, $\lambda_2 = 700 \text{ nm}$
 $\Delta\nu_1 = 10^{-1} \text{ GHz}$, $\Delta\nu_2 = 10^{-3} \text{ GHz}$

$$\nu = \frac{c}{\lambda} \Rightarrow \Delta\nu = -\frac{c}{\lambda^2} \Delta\lambda, \text{ where } \Delta\lambda \text{ is line width.}$$

$$\Rightarrow \Delta\lambda = -\frac{\Delta\nu \lambda^2}{c} \text{ hence, } \frac{\Delta\lambda_1}{\Delta\lambda_2} = \frac{\Delta\nu_1}{\Delta\nu_2} \left(\frac{\lambda_1}{\lambda_2} \right)^2.$$

$$\Rightarrow \frac{\Delta\lambda_1}{\Delta\lambda_2} = \frac{10^{-1}}{10^{-3}} \left(\frac{600}{700} \right) = \frac{3600}{49} \cong 75$$

Correct answer is (c)

39. Let (V, A) and (V', A') denote two sets of scalar and vector potentials, and ψ a scalar function. Which of the following transformations leave the electric and magnetic fields (and hence Maxwell's equations) unchanged?

- (a) $A' = A + \nabla\psi$ and $V' = V - \frac{\partial\psi}{\partial t}$ (b) $A' = A - \nabla\psi$ and $V' = V + 2\frac{\partial\psi}{\partial t}$
(c) $A' = A + \nabla\psi$ and $V' = V + \frac{\partial\psi}{\partial t}$ (d) $A' = A - 2\nabla\psi$ and $V' = V - \frac{\partial\psi}{\partial t}$

Soln. We can choose different sets of (V, A) so that fields remains unchanged after transformation.

$$\text{If } \vec{A}' = \vec{A} + \nabla\psi \text{ then } V' = V - \frac{\partial\psi}{\partial t}$$

Correct answer is (a)

40. Consider the melting transition of ice into water at constant pressure. Which of the following thermodynamic quantities does not exhibit a discontinuous change across the phase transition?
(a) internal energy (b) Helmholtz free energy
(c) Gibbs free energy (d) entropy
41. Two different thermodynamic systems are described by the following equations of state:

$$\frac{1}{T^{(1)}} = \frac{3RN^{(1)}}{2U^{(1)}} \text{ and } \frac{1}{T^{(2)}} = \frac{5RN^{(2)}}{2U^{(2)}} \text{ where } T^{(1,2)}, N^{(1,2)} \text{ and } U^{(1,2)} \text{ are respectively, the temperatures, the mole numbers and the internal energies of the two systems, and } R \text{ is the gas constant. Let } U_{\text{tot}} \text{ denote the total energy when these two systems are put in contact and attain thermal equilibrium.}$$

The ratio $\frac{U^{(1)}}{U_{\text{tot}}}$ is

- (a) $\frac{5N^{(2)}}{3N^{(1)} + 5N^{(2)}}$ (b) $\frac{3N^{(1)}}{3N^{(1)} + 5N^{(2)}}$ (c) $\frac{N^{(1)}}{N^{(1)} + N^{(2)}}$ (d) $\frac{N^{(2)}}{N^{(1)} + N^{(2)}}$

Soln. Let $\bar{T}^{(1)}$ & $\bar{T}^{(2)}$ and $\bar{U}^{(1)}$ & $\bar{U}^{(2)}$ be the temperature and internal energies of first and second system respectively. When both systems are in thermal equilibrium.

When both of the systems are in thermal contact then energy exchange takes place as

$$U^{(1)} + U^{(2)} = \bar{U}^{(1)} + \bar{U}^{(2)} = U_{\text{total}} = \text{constant}$$

When both of the systems attain thermal equilibrium their temperatures become equal.

$$\bar{T}^{(1)} = \bar{T}^{(2)} \quad (\text{But } T^{(1)} \neq \bar{T}^{(1)} \text{ \& } T^{(2)} \neq \bar{T}^{(2)})$$

Then,

$$\begin{aligned} \frac{U^{(1)}}{U_{tot}} &= \frac{\frac{3}{2}N^{(1)}RT^{(1)}}{\frac{3}{2}N^{(1)}RT^{(1)} + \frac{5}{2}N^{(2)}RT^{(2)}} = \frac{3N^{(1)}T^{(1)}}{3N^{(1)}T^{(1)} + 5N^{(2)}T^{(2)}} = \frac{3N^{(1)}T^{(1)}}{3N^{(1)}\bar{T}^{(1)} + 5N^{(2)}\bar{T}^{(2)}} \\ &= \frac{3N^{(1)}T^{(1)}}{\bar{T}^{(1)}(3N^{(1)} + 5N^{(2)})} \quad (\text{as } \bar{T}^{(1)} = \bar{T}^{(2)}) \end{aligned}$$

$$\frac{U^{(1)}}{U_{tot}} = \frac{T^{(1)}}{\bar{T}^{(1)}} \left(\frac{3N^{(1)}}{3N^{(1)} + 5N^{(2)}} \right) \quad \text{this must be the correct answer.}$$

If $U^{(1)}$ is the energy of first system in thermal equilibrium then

$$\frac{\bar{U}^{(1)}}{U_{tot}} = \frac{\bar{T}^{(1)}}{\bar{T}^{(1)}} \left(\frac{3N^{(1)}}{3N^{(1)} + 5N^{(2)}} \right) \Rightarrow \frac{\bar{U}^{(1)}}{U_{tot}} = \frac{3N^{(1)}}{3N^{(1)} + 5N^{(2)}}$$

Correct answer is (b)

42. The speed v of the molecules of mass m of an ideal gas obeys Maxwell's velocity distribution law at an equilibrium temperature T . Let (v_x, v_y, v_z) denote the components of the velocity and k_B the Boltzmann constant. The average value of $(\alpha v_x - \beta v_y)^2$ where α and β are constants, is
- (a) $(\alpha^2 - \beta^2)k_B T / m$ (b) $(\alpha^2 + \beta^2)k_B T / m$ (c) $(\alpha + \beta)k_B T / m$ (d) $(\alpha - \beta)^2 k_B T / m$

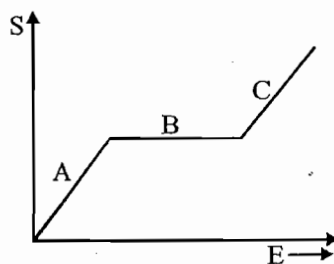
Soln. $\langle (\alpha v_x - \beta v_y)^2 \rangle = \alpha^2 \langle v_x^2 \rangle + \beta^2 \langle v_y^2 \rangle - 2\alpha\beta \langle v_x \rangle \langle v_y \rangle$

$$= \alpha^2 \frac{kT}{m} + \beta^2 \frac{kT}{m} - 2\alpha\beta \frac{kT}{m} = (\alpha^2 + \beta^2) \frac{kT}{m}$$

$$\left[\langle v_x^2 \rangle = \langle v_y^2 \rangle = \frac{kT}{m}, \langle v_x \rangle = \langle v_y \rangle = 0 \right]$$

Correct answer is (b)

43. The entropy S of a thermodynamic system as a function of energy E is given by the following graph



The temperatures of the phases A, B and C, denoted by T_A , T_B and T_C , respectively, satisfy the following inequalities:

- (a) $T_C > T_B > T_A$ (b) $T_A > T_C > T_B$ (c) $T_B > T_C > T_A$ (d) $T_B > T_A > T_C$

Soln. $\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)$

$$\left(\frac{\partial S}{\partial E} \right)_A > \left(\frac{\partial S}{\partial E} \right)_E > \left(\frac{\partial S}{\partial E} \right)_B$$

$$T_B > T_C > T_A$$

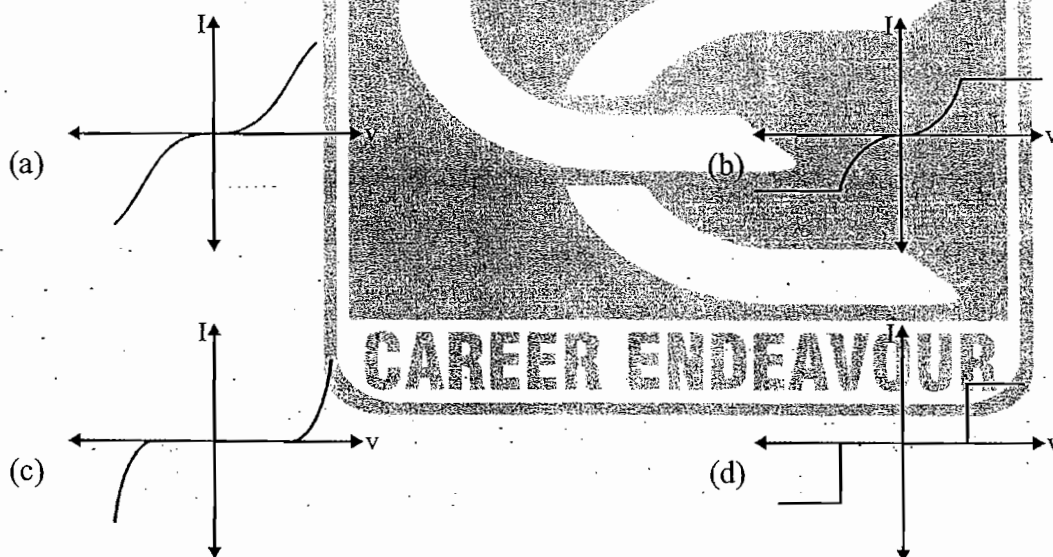
Correct answer is (c)

44. The physical phenomenon that cannot be used for memory storage applications is
- large variation in magnetoresistance as a function of applied magnetic field
 - variation in magnetization of a ferromagnet as a function of applied magnetic field
 - variation in polarization of a ferroelectric as a function of applied electric field
 - variation in resistance of a metal as a function of applied electric field

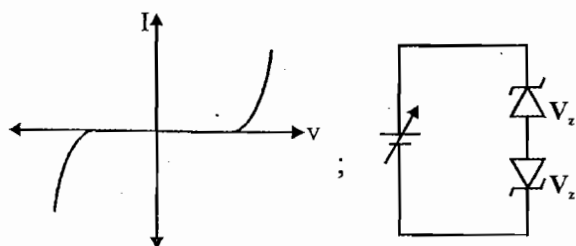
Soln. Variation in resistance of metal as a function of electric field cannot store the memory.

Correct answer is (d)

45. Two identical Zener diodes are placed back to back in series and are connected to a variable DC power supply. The best representation of the I-V characteristics of the circuit is



Soln. The best representation of the I-V characters of the circuit is $\rightarrow$ (with DC supply)



Correct answer is (c)

Part-C

46. A pendulum consists of a ring of mass M and radius R suspended by a massless rigid rod of length l attached to its rim. When the pendulum oscillates in the plane of the ring, the time period of oscillation is

- (a) $2\pi\sqrt{\frac{l+R}{g}}$ (b) $\frac{2\pi}{\sqrt{g}}(l^2 + R^2)^{1/4}$
 (c) $2\pi\sqrt{\frac{2R^2 + 2Rl + l^2}{g(R+l)}}$ (d) $\frac{2\pi}{\sqrt{g}}(2R^2 + 2Rl + l^2)^{1/4}$

Soln. This is a compound pendulum

$$\therefore T = 2\pi\sqrt{\frac{I}{mg\ell_0}}$$

Here, $I = \text{M.I. about point of suspension}$

$\ell_0 = \text{distance of point of suspension from centre of gravity.}$

Here,

$$I = I_c + md^2$$

$$I = mR^2 + m(\ell + R)^2$$

$$I = mR^2 + m(\ell + R)^2$$

$$\therefore T = 2\pi\sqrt{\frac{mR^2 + m(\ell + R)^2}{mg(R + \ell)}} = 2\pi\sqrt{\frac{2R^2 + \ell^2 + 2R\ell}{g(R + \ell)}}$$

Correct answer is (c)

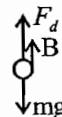
47. Spherical particles of a given material of density ρ are released from rest inside a liquid medium of lower density. The viscous drag force may be approximated by the Stoke's law, i.e., $F_d = 6\pi\eta Rv$, where η is the viscosity of the medium, R the radius of a particle and v its instantaneous velocity. If $\tau(m)$ is the time taken by a particle of mass m to reach half its terminal velocity, then the ratio $\tau(8m)/\tau(m)$ is

- (a) 8 (b) 1/8 (c) 4 (d) 1/4

Soln. Each particle has same density but different radii and masses. We have to calculate time of fall in terms of mass of particle therefore we will write our equations explicitly in terms of mass and we will remove radius from our equations wherever it appears.

Drag force on particles

$$F_d = 6\pi\eta Rv, \quad \text{Mass of a particle } m = \rho \cdot \frac{4}{3}\pi R^3$$



$$\therefore R = \left(\frac{3m}{4\pi\rho}\right)^{1/3} \quad \therefore F_d = 6\pi\eta \left(\frac{3m}{4\pi\rho}\right)^{1/3} v = Km^{1/3}v, \text{ where } K = 6\pi\eta \left(\frac{3}{4\pi\rho}\right)^{1/3}$$

If B is buoyancy force, then equation of motion of a particle is

$$m \frac{dv}{dt} = mg - F_d - B \quad \text{where } B = \frac{4}{3}\pi R^3 \sigma g = \frac{4}{3}\pi R^3 \rho g \cdot \frac{\sigma}{\rho} = mg \frac{\sigma}{\rho}, \quad \sigma = \text{density of medium}$$

$$\therefore m \frac{dv}{dt} = mg - Km^{1/3}v - mg \frac{\sigma}{\rho}, \quad m \frac{dv}{dt} = mg \left(1 - \frac{\sigma}{\rho}\right) - Km^{1/3}v$$



$$\text{or } \frac{dv}{dt} = g \left(1 - \frac{\sigma}{\rho} \right) - Km^{-2/3} v \quad \dots (a)$$

when terminal velocity is reached $\frac{dv}{dt} = 0$

$$\therefore 0 = g \left(1 - \frac{\sigma}{\rho} \right) - Km^{-2/3} v_t \quad \therefore v_t = \frac{g \left(1 - \frac{\sigma}{\rho} \right)}{Km^{-2/3}} \quad \dots (b)$$

Now, if τ be the time to reach half the terminal velocity then from (a)

$$\int_0^{v_t/2} \frac{dv}{g \left(1 - \frac{\sigma}{\rho} \right) - Km^{-2/3} v} = \int_0^{\tau} dt, \quad \therefore -\frac{1}{Km^{-2/3}} \ln \left[\frac{g \left(1 - \frac{\sigma}{\rho} \right) - \frac{Km^{-2/3} v_t}{2}}{g \left(1 - \frac{\sigma}{\rho} \right)} \right] = \tau$$

using value of v_t from (b) we get, $\tau = \frac{m^{2/3}}{K} \ln 2$ or $\tau(m) = \frac{m^{2/3} \ln 2}{K}$

$$\therefore \tau(8m) / \tau(m) = (8m)^{2/3} / m^{2/3} = 4$$

Correct answer is (c)

48. A system of N classical non-interacting particles, each of mass m , is at a temperature T and is confined by the external potential $V(r) = \frac{1}{2} Ar^2$ (where A is a constant) in three dimensions. The internal energy of the system is

(a) $3Nk_B T$

(b) $\frac{3}{2} Nk_B T$

(c) $N(2mA)^{3/2} k_B T$

(d) $N \sqrt{\frac{A}{m}} \ln \left(\frac{k_B T}{m} \right)$

Soln. The contribution of thermal energy with each quadratic term in hamiltonian is $\frac{1}{2} kT$.

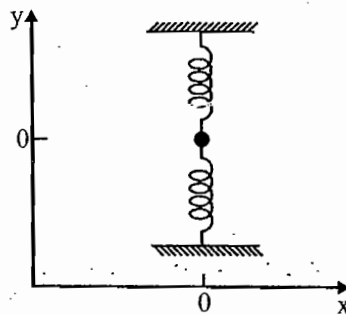
The hamiltonian of a classical particle $\mu = \frac{p^2}{2m} + \frac{1}{2} Ar^2$

IN 3-D, hamiltonian has 6 quadratic term. So, the internal energy of a particle is $6 \times \frac{1}{2} kT = 3kT$

The internal energy of a system of N particles = $3NkT$.

Correct answer is (a)

49. Consider a particle of mass m attached to two identical springs each of length l and spring constant k (see the figure below). The equilibrium configuration is the one where the springs are unstretched. There are no other external forces on the system. If the particle is given a small displacement along the x -axis, which of the following describes the equation of motion for small oscillations?



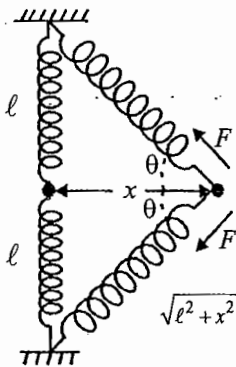
(a) $m\ddot{x} + \frac{kx^3}{l^2} = 0$

(b) $m\ddot{x} + kx = 0$

(c) $m\ddot{x} + 2kx = 0$

(d) $m\ddot{x} + \frac{kx^2}{l} = 0$

Soln.



Let x be the displacement of particle along x -axis.

Therefore, net force on the particle along ' x ' axis $= -2F \cos \theta$

Where, $F = k \times \text{elongation} = k(\sqrt{l^2 + x^2} - l)$

Therefore, using Newton's law we get {Net force $= m\ddot{x}$ }

$$-2F \cos \theta = m\ddot{x}$$

$$\Rightarrow -2k \left(\sqrt{l^2 + x^2} - l \right) \cdot \frac{x}{\sqrt{l^2 + x^2}} = m\ddot{x}$$

$$\Rightarrow -2kx \left[1 - l \left(l^2 + x^2 \right)^{-1/2} \right] = m\ddot{x} \Rightarrow -2kx \left[1 - \left(1 + \frac{x^2}{l^2} \right)^{-1/2} \right] = m\ddot{x}$$

Using binomial expansion {keep only upto x^2 term}

We get, $-2kx \cdot \frac{x^2}{2l^2} = m\ddot{x} \quad \therefore \quad m\ddot{x} + \frac{kx^3}{l^2} = 0$

Correct answer is (a).

50. If $\psi(x) = A \exp(-x^4)$ is the eigenfunction of a one dimensional Hamiltonian with eigenvalue $E = 0$, the potential $V(x)$ (in units where $\hbar = 2m = 1$) is
- (a) $12x^2$ (b) $16x^6$ (c) $16x^6 + 12x^2$ (d) $16x^6 - 12x^2$

Soln. Given : $\psi(x) = A \exp(-x^4)$ is an eigenfunction of a 1-D hamiltonian with eigenvalue $E = 0$

From 1-D Schrodinger equation, $-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = 0$

$$\Rightarrow -\frac{\hbar^2}{2m} \left[A \exp(-x^4) \{16x^6 - 12x^2\} \right] + V\psi = 0$$

Taking $\hbar = 1, 2m = 1 \Rightarrow -\psi \{16x^6 - 12x^2\} + V\psi = 0$

$$\Rightarrow V(x) = 16x^6 - 12x^2$$

Correct answer is (d)

51. The electric field of an electromagnetic wave is given by $\vec{E} = E_0 \cos[\pi(0.3x + 0.4y - 1000t)]\hat{k}$.

The associated magnetic field $\vec{B}$ is

- (a) $10^{-3} E_0 \cos[\pi(0.3x + 0.4y - 1000t)]\hat{k}$
 (b) $10^{-4} E_0 \cos[\pi(0.3x + 0.4y - 1000t)](4\hat{i} - 3\hat{j})$
 (c) $E_0 \cos[\pi(0.3x + 0.4y - 1000t)](0.3\hat{i} + 0.4\hat{j})$
 (d) $10^2 E_0 \cos[\pi(0.3x + 0.4y - 1000t)](3\hat{i} + 4\hat{j})$

Soln. Electric field of an e.m. wave is given by

$$\vec{E} = E_0 \cos[\pi(0.3x + 0.4y - 1000t)]\hat{k}$$

Wave vector $\vec{k}$ is given by $\vec{k} = \pi(0.3\hat{i} + 0.4\hat{j})$

$$\text{Hence, } \vec{B} = \frac{(\vec{k} \times \vec{E})}{\omega} = \frac{[\pi(0.3\hat{i} + 0.4\hat{j}) \times \hat{k}] E_0 \cos[\pi(0.3x + 0.4y - 1000t)]}{1000\pi}$$

$$= 10^{-3} (-0.3\hat{i} + 0.4\hat{j}) E_0 \cos[\pi(0.3x + 0.4y - 1000t)]\hat{k}$$

Correct answer is (b)

52. The energy of an electron in a band as a function of its wave vector $\mathbf{k}$ is given by

$E(k) = E_0 - B(\cos k_x a + \cos k_y a + \cos k_z a)$, where E_0, B and a are constants. The effective mass of the electron near the bottom of the band is

- (a) $\frac{2\hbar^2}{3Ba^2}$ (b) $\frac{\hbar^2}{3Ba^2}$ (c) $\frac{\hbar^2}{2Ba^2}$ (d) $\frac{\hbar^2}{Ba^2}$

Soln. $E(k) = E_0 - B(\cos k_x a + \cos k_y a + \cos k_z a)$

Near the bottom of the band

$$E(k) = E_0 - B \left[\left(1 - \frac{k_x^2 a^2}{2} \right) + \left(1 - \frac{k_y^2 a^2}{2} \right) + \left(1 - \frac{k_z^2 a^2}{2} \right) \right] = E_0 - B \left[3 - \frac{a^2}{2} (k_x^2 + k_y^2 + k_z^2) \right]$$

$$E(k) = E_0 - 3B + B \frac{a^2}{2} k^2 \Rightarrow \frac{dE}{dk} = Ba^2 k \Rightarrow \frac{d^2 E}{dk^2} = Ba^2$$

Effective mass, $m^* = \frac{\hbar^2}{Ba^2}$

Correct answer is (d)

53. A DC voltage V is applied across a Josephson junction between two superconductors with a phase difference ϕ_0 . If I_0 and k are constants that depend on the properties of the junction, the current flowing through it has the form

(a) $I_0 \sin \left(\frac{2eVt}{\hbar} + \phi_0 \right)$ (b) $kV \sin \left(\frac{2eVt}{\hbar} + \phi_0 \right)$ (c) $kV \sin \phi_0$ (d) $I_0 \sin \phi_0 + kV$

Soln. On the application of DC voltage, AC current flows across the junction and that is given by

$$I = I_0 \sin \left(\frac{2eVt}{\hbar} + \phi_0 \right)$$

Correct answer is (a)

54. Consider the following ratios of the partial decay widths $R_1 = \frac{\Gamma(\rho^+ \rightarrow \pi^+ + \pi^0)}{\Gamma(\rho^- \rightarrow \pi^- + \pi^0)}$ and

$$R_2 = \frac{\Gamma(\Delta^{++} \rightarrow \pi^+ + p)}{\Gamma(\Delta^- \rightarrow \pi^- + n)}$$

If the effects of electromagnetic and weak interactions are neglected, then

R_1 and R_2 are, respectively,

- (a) 1 and $\sqrt{2}$ (b) 1 and 2 (c) 2 and 1 (d) 1 and 1

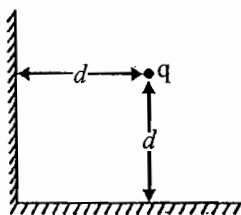
55. The intrinsic electric dipole moment of a nucleus ${}^A_Z X$

- (a) increases with Z , but independent of A (b) decreases with Z , but independent of A
(c) is always zero (d) increases with Z and A

56. According to the shell model, the total angular momentum (in units of $\hbar$) and the parity of the ground state of the ${}^7_3\text{Li}$ nucleus is

- (a) $\frac{3}{2}$ with negative parity (b) $\frac{3}{2}$ with positive parity
(c) $\frac{1}{2}$ with positive parity (d) $\frac{7}{2}$ with negative parity

57. A point charge q is placed symmetrically at a distance d from two perpendicularly placed grounded conducting infinite plates as shown in the figure. The net force on the charge (in units of $1/4\pi\epsilon_0$) is



- (a) $\frac{q^2}{8d^2}(2\sqrt{2}-1)$ away from the corner
 (b) $\frac{q^2}{8d^2}(2\sqrt{2}-1)$ towards the corner
 (c) $\frac{q^2}{2\sqrt{2}d^2}$ towards the corner
 (d) $\frac{3q^2}{8d^2}$ away from the corner

Soln. Force on charge $+q$ will be attractive type

$$\vec{F}_1 = +\frac{q^2}{4\pi\epsilon_0(4d^2)}(-\hat{x})$$

$$\vec{F}_2 = +\frac{q^2}{4\pi\epsilon_0(4d^2)}(-\hat{y})$$

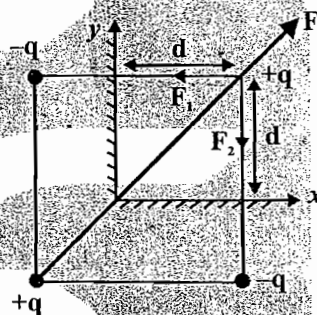
$$\vec{F}_3 = +\frac{q^2}{4\pi\epsilon_0(8d^2)}\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)$$

$$\text{Net force, } F = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{q^2}{4\pi\epsilon_0(8d^2)}\left[-2\hat{x}-2\hat{y}+\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right]$$

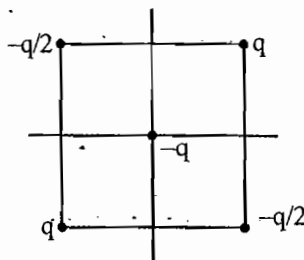
$$= \frac{q^2}{4\pi\epsilon_0(8d^2)}\left[\left(\frac{-2\sqrt{2}+1}{\sqrt{2}}\right)\hat{x}+\left(\frac{-2\sqrt{2}+1}{\sqrt{2}}\right)\hat{y}\right]$$

$$|F| = \frac{q^2}{4\pi\epsilon_0(8d^2)}(2\sqrt{2}-1) \quad \text{Towards the corner.}$$

Correct answer is (b)



58. Let four point charges q , $-q/2$, q and $-q/2$ be placed at the vertices of a square of side a . Let another point charge $-q$ be placed at the centre of the square (see the figure).



Let $V(r)$ be the electrostatic potential at a point P at a distance $r \gg a$ from the centre of the square. Then $V(2r) / V(r)$ is

- (a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

Soln. Monopole and dipole moment of the charge configuration is zero. Hence, potential at large

distances $r \gg a$ varies as

$$V(r) \propto \frac{1}{r^3}$$

Hence,
$$\frac{V(2r)}{V(r)} = \frac{1}{8r^3} \times \frac{r^3}{1} = \frac{1}{8}$$

Correct answer is (d).

59. Let A and B be two vectors in three-dimensional Euclidean space. Under rotation, the tensor product $T_{ij} = A_i B_j$

- (a) reduces to a direct sum of three 3-dimensional representations
 (b) is an irreducible 9-dimensional representation
 (c) reduces to a direct sum of a 1-dimensional, a 3-dimensional and a 5-dimensional irreducible representations
 (d) reduces to a direct sum of a 1-dimensional and an 8-dimensional irreducible representation

60. Fourier transform of the derivative of the Dirac δ -function, namely $\delta'(x)$, is proportional to
 (a) 0 (b) 1 (c) sink (d) ik

Soln. Fourier transform of $\delta'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta'(x) e^{ikx} dx$

Using the property of Dirac delta function
$$\int_{-\infty}^{\infty} f(x) \delta'(x-a) dx = -f'(a)$$

So,
$$F[\delta'(x)] = -\frac{1}{\sqrt{2\pi}}(ik) \Rightarrow F[\delta'(x)] \propto ik$$

Correct answer is (d)

61. A particle is in the ground state of an infinite square well potential given by,

$$V(x) = \begin{cases} 0 & \text{for } -a \leq x \leq a \\ \infty & \text{otherwise} \end{cases}$$

The probability to find the particle in the interval between $-\frac{a}{2}$ and $\frac{a}{2}$ is

- (a) $\frac{1}{2}$ (b) $\frac{1}{2} + \frac{1}{\pi}$ (c) $\frac{1}{2} - \frac{1}{\pi}$ (d) $\frac{1}{\pi}$

Soln. Given : $V(x) = \begin{cases} 0 & \text{for } -a \leq x \leq a \\ \infty & \text{otherwise} \end{cases}$

Symmetrical infinite potential well of width '2a'

Probability of finding the particle in the interval between $-\frac{a}{2}$ to $\frac{a}{2}$ is

$$P = \int_{-a/2}^{a/2} \psi_1^* \psi_1 dx = \frac{1}{a} \int_{-a/2}^{a/2} \cos^2\left(\frac{\pi x}{2a}\right) dx = \frac{1}{2a} \int_{-a/2}^{a/2} \left[1 + \cos\left(\frac{\pi x}{a}\right)\right] dx$$

$$= \frac{1}{2a} \left[x + \frac{a}{\pi} \sin\left(\frac{\pi x}{a}\right) \right]_{-a/2}^{a/2} = \frac{1}{2a} \left[a + \frac{2a}{\pi} \right] = \frac{1}{2} + \frac{1}{\pi}$$

Correct answer is (b)

62. The expectation value of the x-component of the orbital angular momentum L_x in the state

$\psi = \frac{1}{5} [3\psi_{2,1,-1} + \sqrt{5}\psi_{2,1,0} - \sqrt{11}\psi_{2,1,1}]$ (where ψ_{nlm} are the eigenfunctions in usual notation), is

- (a) $-\frac{\hbar\sqrt{10}}{25}(\sqrt{11}-3)$ (b) 0
(c) $\frac{\hbar\sqrt{10}}{25}(\sqrt{11}+3)$ (d) $\hbar\sqrt{2}$

Soln. Given : $\psi = \frac{1}{5} [3\psi_{2,1,-1} + \sqrt{5}\psi_{2,1,0} - \sqrt{11}\psi_{2,1,1}]$

$$\langle L_x \rangle = \langle \psi | L_x | \psi \rangle$$

$$= \frac{1}{5} [3\langle \psi_{2,1,-1} | + \sqrt{5}\langle \psi_{2,1,0} | - \sqrt{11}\langle \psi_{2,1,1} |] \left[\frac{1}{2}(L_+ + L_-) \right]$$

$$\left[\frac{3}{5} |\psi_{2,1,-1}\rangle + \frac{\sqrt{5}}{5} |\psi_{2,1,0}\rangle - \frac{\sqrt{11}}{5} |\psi_{2,1,1}\rangle \right]$$

Now, we know, $\hat{L}_+ \psi_{nlm} = \sqrt{(\ell-m)(\ell+m+1)} \hbar \psi_{n\ell m+1}$

$$\hat{L}_- \psi_{nlm} = \sqrt{(\ell+m)(\ell-m+1)} \hbar \psi_{n\ell m-1}$$

$$\Rightarrow \langle L_x \rangle = \left[\frac{3}{5} \langle \psi_{2,1,-1} | + \frac{\sqrt{5}}{5} \langle \psi_{2,1,0} | - \frac{\sqrt{11}}{5} \langle \psi_{2,1,1} | \right]$$

$$\frac{1}{2} \left[\frac{3}{5} \sqrt{2} \hbar |\psi_{2,1,0}\rangle + \frac{\sqrt{5}}{5} \sqrt{2} \hbar |\psi_{2,1,1}\rangle + \frac{\sqrt{5}}{5} \sqrt{2} \hbar |\psi_{2,1,-1}\rangle - \frac{\sqrt{11}}{5} \sqrt{2} \hbar |\psi_{2,1,0}\rangle \right]$$

$$\Rightarrow \langle L_x \rangle = \frac{1}{2} \frac{\sqrt{5}}{5} \cdot \sqrt{2} \hbar \left(\frac{3}{5} - \frac{\sqrt{11}}{5} \right) + \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{\sqrt{5}}{5} \sqrt{2} \hbar - \frac{\sqrt{11}}{5} \cdot \frac{1}{2} \cdot \frac{\sqrt{5}}{5} \sqrt{2} \hbar$$

$$= \frac{3\sqrt{5}}{25} \sqrt{2} \hbar - \frac{\sqrt{11}\sqrt{5}}{25} \sqrt{2} \hbar = -\frac{\hbar\sqrt{10}}{25} (\sqrt{11} - 3)$$

Correct answer is (a)

63. A particle is prepared in a simultaneous eigenstate of L^2 and L_z . If $\ell(\ell+1)\hbar^2$ and $m\hbar$ are respectively the eigenvalues of L^2 and L_z , then the expectation value $\langle L_x^2 \rangle$ of the particle in this state satisfies

(a) $\langle L_x^2 \rangle = 0$

(b) $0 \leq \langle L_x^2 \rangle \leq \ell^2 \hbar^2$

(c) $0 \leq \langle L_x^2 \rangle \leq \frac{\ell(\ell+1)\hbar^2}{3}$

(d) $\frac{\hbar^2}{2} \leq \langle L_x^2 \rangle \leq \frac{\ell(\ell+1)\hbar^2}{2}$

Soln. $\langle \hat{L}_x^2 + \hat{L}_y^2 \rangle = \langle \hat{L}^2 \rangle - \langle \hat{L}_z^2 \rangle = \ell(\ell+1)\hbar^2 - m^2\hbar^2$

Since, $\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle$ then $\langle \hat{L}_x^2 \rangle = \frac{1}{2} [\ell(\ell+1)\hbar^2 - m^2\hbar^2]$

For a given ℓ , m varies from $-\ell$ to ℓ .

$\langle \hat{L}_x^2 \rangle_{\max}$ occurs when $m=0 \Rightarrow \langle \hat{L}_x^2 \rangle_{\max} = \frac{1}{2} \ell(\ell+1)\hbar^2$

$\langle \hat{L}_x^2 \rangle_{\min}$ occurs when $m=\pm\ell \Rightarrow \langle \hat{L}_x^2 \rangle_{\min} = \frac{1}{2} \ell\hbar^2$

Correct answer is (d).

64. If the electrostatic potential $V(r, \theta, \phi)$ in a charge free region has the form $V(r, \theta, \phi) = f(r) \cos \theta$, then the functional form of $f(r)$ (in the following a and b are constants) is

(a) $ar^2 + \frac{b}{r}$

(b) $ar + \frac{b}{r^2}$

(c) $ar + \frac{b}{r}$

(d) $a \ln \left(\frac{r}{b} \right)$

Soln. Potential is given by $V(r, \theta, \phi) = f(r) \cos \theta$

Laplace equation is given by

$$\nabla^2 V(r, \theta, \phi) = 0 \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left(\sin \theta \frac{\partial v}{\partial \theta} \right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r} \left(r^2 f'(r) \cos \theta \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta f(r) (-\sin \theta) \right) = 0$$

$$\Rightarrow \cos \theta \left[r^2 f''(r) + 2rf'(r) \right] - \frac{2 \sin \theta \cos \theta f(r)}{\sin \theta} = 0$$

$$\Rightarrow \boxed{r^2 f''(r) + 2rf'(r) - 2f(r) = 0} \quad \dots (i)$$

The value of $f(r)$ in option (b) satisfy the equation (i).

Correct answer is (b)

65. If $\mathbf{A} = \hat{i}yz + \hat{j}xz + \hat{k}xy$, then the integral $\oint_C \mathbf{A} \cdot d\mathbf{l}$ (where C is along the perimeter of a rectangular area bounded by $x = 0, x = a$ and $y = 0, y = b$) is

- (a) $\frac{1}{2}(a^3 + b^3)$ (b) $\pi(ab^2 + a^2b)$ (c) $\pi(a^3 + b^3)$ (d) 0

Soln. $\vec{A} = yz\hat{i} + xz\hat{j} + xy\hat{k} \Rightarrow \vec{\nabla} \times \vec{A} = 0$ i.e. field is conservative.

Since, line integral of a conservative field along a closed path is zero, then $\oint_C \vec{A} \cdot d\vec{\ell} = 0$

Correct answer is (d)

66. Consider an $n \times n (n > 1)$ matrix A , in which A_{ij} is the product of the indices i and j (namely $A_{ij} = ij$). The matrix A
- (a) has one degenerate eigenvalue with degeneracy $(n-1)$
- (b) has two degenerate eigenvalues with degeneracies 2 and $(n-2)$
- (c) has one degenerate eigenvalue with degeneracy n
- (d) does not have any degenerate eigenvalue

Soln. Given : $A_{ij} = ij$. So, for a 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

Since, the second row and third row is multiple of first row and the same is true for columns, then eigenvalue of the matrix will be $\text{Tr}(A)$, 0, 0 i.e. 14, 0, 0

'0' is a degenerate eigenvalue with degeneracy '2'

In general, matrix A (order 'n') has a degenerate eigenvalue with degeneracy 'n-1'.

Correct answer is (a)

67. A child makes a random walk on a square lattice of lattice constant a taking a step in the north, east, south, or west directions with probabilities 0.255, 0.255, 0.245, and 0.245, respectively. After a large number of steps, N , the expected position of the child with respect to the starting point is at a distance

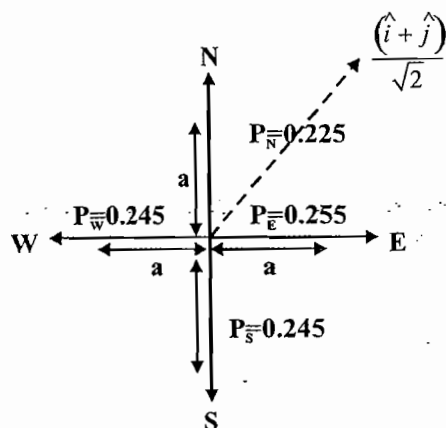
- (a) $\sqrt{2} \times 10^{-2} Na$ in the north-east direction (b) $\sqrt{2N} \times 10^{-2} a$ in the north-east direction
- (c) $2\sqrt{2} \times 10^{-2} Na$ in the south-east direction (d) 0

Soln. Single step length = a

After a single step, the expected value of position of the child.

$$\langle \bar{x} \rangle_1 = \sum_i P_i \bar{x}_i$$

$$\Rightarrow \langle \vec{x} \rangle_1 = 0.255(a\hat{i}) + 0.255(a\hat{j}) + 0.245(-a\hat{i}) + 0.245(-a\hat{j})$$



$$\Rightarrow \langle \vec{x} \rangle_1 = 0.01a(\hat{i} + \hat{j})$$

After 'n' steps, $\langle \vec{x} \rangle_N = 0.01 Na(\hat{i} + \hat{j})$

Expected position $\langle \vec{x} \rangle_N = \sqrt{2} \times 10^{-2} Na$ (NORTH-EAST DIRECTION)

Correct answer is (a)

68. A Carnot cycle operates as a heat engine between two bodies of equal heat capacity until their temperatures become equal. If the initial temperatures of the bodies are T_1 and T_2 , respectively, and $T_1 > T_2$ then their common final temperature is

(a) T_1^2 / T_2

(b) T_2^2 / T_1

(c) $\sqrt{T_1 T_2}$

(d) $\frac{1}{2}(T_1 + T_2)$

Soln. Carnot cycle is reversible cycle so the entropy change must be zero.

$$\Delta S = 0$$

Let the equilibrium temperature be T then

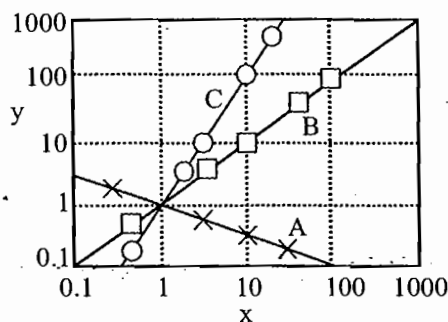
$$\Delta S = C \int_{T_1}^T \frac{dT}{T} + C \int_{T_2}^T \frac{dT}{T} = 0$$

$$\ln\left(\frac{T}{T_1}\right) + \ln\left(\frac{T}{T_2}\right) = 0$$

$$\ln\left(\frac{T^2}{T_1 T_2}\right) = 0 \quad \Rightarrow T^2 = T_1 T_2 \quad \Rightarrow T = \sqrt{T_1 T_2}$$

Correct answer is (c)

69. Three sets of data A , B and C from an experiment, represented by $\times$, $\square$ and $\circ$, are plotted on a log-log scale. Each of these are fitted with straight lines as shown in the figure.



The functional dependence $y(x)$ for the sets A , B and C are, respectively

(a) $\sqrt{x}$, x and x^2

(b) $-\frac{x}{2}$, x and $2x$

(c) $\frac{1}{x^2}$, x and x^2

(d) $\frac{1}{\sqrt{x}}$, x and x^2

Soln. Plot of Data A : $\frac{3\pi}{4} < \theta < \pi \Rightarrow m = \tan \theta$ is between 0 and -1 .

Plot of Data B : $\theta = \frac{\pi}{4} \Rightarrow m = \tan \theta = 1$

Plot of Data C : $\frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow m = \tan \theta$ is > 1

Since, datas are plotted on log-log scale and all passes through $(1, 1)$ then the straight lines will be of the nature, $\log y = m \log x$

Correct answer is (d).

70. A sample of Si has electron and hole mobilities of 0.13 and $0.05 \text{ m}^2/\text{V-s}$ respectively at 300K . It is doped with P and Al with doping densities of $1.5 \times 10^{21}/\text{m}^3$ and $2.5 \times 10^{21}/\text{m}^3$ respectively. The conductivity of the doped Si sample at 300K is

(a) $8\Omega^{-1}\text{m}^{-1}$

(b) $32\Omega^{-1}\text{m}^{-1}$

(c) $20.8\Omega^{-1}\text{m}^{-1}$

(d) $83.2\Omega^{-1}\text{m}^{-1}$

Soln. Si doped with P and Al.

As doping of Al $>$ doping of P.

Hence, semiconductor will behave as p-type semiconductor.

$$\text{Majority carrier concentration} (2.5 \times 10^{21} - 1.5 \times 10^{21}) = 1 \times 10^{21}/\text{m}^3$$

$$\text{The conductivity } \sigma = pe \mu_p = 1 \times 10^{21} \times 1.6 \times 10^{-19} = 8\Omega^{-1}\text{m}^{-1}$$

Correct answer is (a)

71. A 4-variable switching function is given by $f = \Sigma(5, 7, 8, 10, 13, 15) + d(0, 1, 2)$, where d is the do-not-care-condition. The minimized form of f in sum of products (SOP) form is

(a) $\overline{A}\overline{C} + \overline{B}\overline{D}$ (b) $\overline{A}\overline{B} + \overline{C}\overline{D}$ (c) $AD + BC$ (d) $\overline{B}\overline{D} + BD$

Soln. 4-variable switching function is given by

$$f(A, B, C, D) = \Sigma(5, 7, 8, 10, 13, 15) + d(0, 1, 2)$$

AB \ CD	00	01	11	10
00	$\overline{X}_0$	$\overline{X}_1$	3	$\overline{X}_2$
01	4	15	17	6
11	12	13	15	14
10	8	9	11	10

There are two Quad.

$$\Rightarrow \overline{D}\overline{B} + DB$$

Correct answer is (d)

72. A perturbation $V_{\text{pert}} = aL^2$ is added to the Hydrogen atom potential. The shift in the energy level of the 2P state, when the effects of spin are neglected up to second order in a , is

(a) 0 (b) $2a\hbar^2 + a^2\hbar^4$ (c) $2a\hbar^2$ (d) $a\hbar^2 + \frac{3}{2}a^2\hbar^4$

Soln. Method-1:

The energy corrected to second order

$$E = E^{(0)} + E^{(1)} + E^{(2)}$$

$$\text{The shift in energy} = E - E^{(0)} = E^{(1)} + E^{(2)}$$

where, $E^{(0)}$ = unperturbed energy

$E^{(1)}$ = first order correction in energy

$E^{(2)}$ = second order correction in energy

$$\text{Perturbation, } V_{\text{pert}} = aL^2$$

2p state of H-atom is 3-fold degenerate. Secular determinant to find first order correction in energy.

$$\begin{vmatrix} \langle 1, +1 | aL^2 | 1, +1 \rangle - E^{(1)} & \langle 1, 0 | aL^2 | 1, 0 \rangle & \langle 1, -1 | aL^2 | 1, -1 \rangle \\ \langle 1, 0 | aL^2 | 1, +1 \rangle & \langle 1, 0 | aL^2 | 1, 0 \rangle - E^{(1)} & \langle 1, 0 | aL^2 | 1, -1 \rangle \\ \langle 1, -1 | aL^2 | 1, +1 \rangle & \langle 1, -1 | aL^2 | 1, 0 \rangle & \langle 1, -1 | aL^2 | 1, -1 \rangle - E^{(1)} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2\hbar^2 - E^{(1)} & 2\hbar^2 & 2\hbar^2 \\ 2\hbar^2 & 2\hbar^2 - E^{(1)} & 2\hbar^2 \\ 2\hbar^2 & 2\hbar^2 & 2\hbar^2 - E^{(1)} \end{vmatrix} = 0 \Rightarrow E^{(1)} = 2\hbar^2$$

The second order correction in energy for degenerate perturbation theory includes matrix element (that has summation over non-deg states). There is no non-deg states, so $E^{(2)} = 0$

The shift in energy = $2\hbar^2$

Correct answer is (c)

Method-2:

Since the energy correction to the all degenerate states are equal. Therefore, we can use the non-degenerate perturbation theory to solve the problem as follows.

Given : $V_{pert} = a\hat{L}^2$ is added to a hydrogen atom potential.

First order correction to energy of the 2p state ($n=2, \ell=1$)

$$E_n^{(1)} = \langle 2P | a\hat{L}^2 | 2P \rangle = a\ell(\ell+1)\hbar^2 = 2a\hbar^2$$

Second order correction to energy of the 2P state ($n=2, \ell=1$)

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n | a\hat{L}^2 | m \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \sum_{m \neq 2P} \frac{|\langle 2P | a\hat{L}^2 | m \rangle|^2}{E_{2P}^{(0)} - E_m^{(0)}} = 0$$

Therefore, the shift in the energy level = $E_n^{(1)} + E_n^{(2)} = 2a\hbar^2$

Correct answer is (c)

73. A gas laser cavity has been designed to operate at $\lambda = 0.5 \mu\text{m}$ with a cavity length of 1m. With this set-up, the frequency is found to be larger than the desired frequency by 100 Hz. The change in the effective length of the cavity required to retune the laser is

- (a) $-0.334 \times 10^{-12} \text{ m}$ (b) $0.334 \times 10^{-12} \text{ m}$
(c) $0.167 \times 10^{-12} \text{ m}$ (d) $-0.167 \times 10^{-12} \text{ m}$

Soln. The separation between cavity mirrors, $d = \frac{m\lambda}{2}$ where 'm' is mode number.

$$\lambda = 0.5 \mu\text{m}, d = 1\text{m}$$

$$\Rightarrow 1 = \frac{m \times 0.5 \times 10^{-6}}{2} \Rightarrow m = \frac{20}{0.5} \times 10^6$$

$$m = 4 \times 10^6$$

Change in length of cavity, $\Delta d = m \left(\frac{\Delta \lambda}{2} \right)$

$$\text{But, } v = \frac{c}{\lambda} \Rightarrow \Delta v = -\frac{c}{\lambda^2} \Delta \lambda$$

$$\Rightarrow \Delta \lambda = -\frac{\Delta v \lambda^2}{c} \Rightarrow \Delta d = -\frac{m \Delta v \lambda^2}{2c}$$

$$\Rightarrow \Delta d = \frac{-0.5}{3} \times 10^{-12} \Rightarrow \boxed{\Delta d = -0.167 \times 10^{-12} \text{ m}}$$

Correct answer is (d)

74. The spectroscopic symbol for the ground state of ${}_{13}\text{Al}$ is ${}^2P_{1/2}$. Under the action of a strong magnetic field (when L-S coupling can be neglected) the ground state energy level will split into
- (a) 3 levels (b) 4 levels (c) 5 levels (d) 6 levels

Soln.

state	m_ℓ	m_s	$m_\ell + 2m_s$
${}^2P_{1/2}$ ($\ell = 1, s = 1/2$)	1	$1/2, -1/2$	2, 0
	0	$1/2, -1/2$	1, -1
	-1	$1/2, -1/2$	0, -2

Out of the given 6 states, 2 states have same energy shift due to strong magnetic field. Therefore, the ground state energy level will split into five levels.

Correct answer is (d)

75. A uniform linear monoatomic chain is modeled by a spring-mass system of masses m separated by nearest neighbor distance a and spring constant $m\omega_0^2$. The dispersion relation for this system is

(a) $\omega(k) = 2\omega_0 \left(1 - \cos\left(\frac{ka}{2}\right) \right)$ (b) $\omega(k) = 2\omega_0 \sin\left(\frac{ka}{2}\right)$

(c) $\omega(k) = 2\omega_0 \sin\left(\frac{ka}{2}\right)$ (d) $\omega(k) = 2\omega_0 \tan\left(\frac{ka}{2}\right)$

Soln. Frequency of monoatomic one dimensional lattice is given by

$$\boxed{\omega(k) = \sqrt{\frac{4c}{m}} \sin\left(\frac{ka}{2}\right)}$$

where 'c' is the force constant (spring constant)

But $\boxed{c = m\omega_0^2}$

$$\omega(k) = \sqrt{\frac{4m\omega_0^2}{m}} \sin\left(\frac{ka}{2}\right) \Rightarrow \boxed{\omega(k) = 2\omega_0 \sin\left(\frac{ka}{2}\right)}$$

Correct answer is (c).



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Part-B

21. One gram of salt is dissolved in water that is filled to a height of 5 cm in a beaker of diameter 10 cm. The accuracy of length measurement is 0.01 cm while that of mass measurement is 0.01 mg. When measuring the concentration C , the fractional error $\Delta C/C$ is
(a) 0.8% (b) 0.14% (c) 0.5% (d) 0.28%

Soln. Correct answer is (a)

22. A system can have three energy levels: $E = 0, \pm \epsilon$. The level $E = 0$ is doubly degenerate, while the others are non-degenerate. The average energy at inverse temperature β is

- (a) $-\epsilon \tanh(\beta\epsilon)$ (b) $\frac{\epsilon(e^{\beta\epsilon} - e^{-\beta\epsilon})}{(1 + e^{\beta\epsilon} + e^{-\beta\epsilon})}$
(c) Zero (d) $-\epsilon \tanh\left(\frac{\beta\epsilon}{2}\right)$

Soln. Average energy of the system

$$\begin{aligned}\langle E \rangle &= \frac{\sum_r g_r E_r e^{-\beta E_r}}{\sum_r g_r e^{-\beta E_r}} \\&= \frac{2 \times 0 + \epsilon e^{-\beta\epsilon} - \epsilon e^{+\beta\epsilon}}{2 + e^{-\beta\epsilon} + e^{+\beta\epsilon}} = \frac{-\epsilon(e^{\beta\epsilon} - e^{-\beta\epsilon})}{2 + e^{-\beta\epsilon} + e^{+\beta\epsilon}} \\&= \frac{-\epsilon 2 \sinh(\beta\epsilon)}{2 + 2 \cosh(\beta\epsilon)} = \frac{-\epsilon 2 \sinh\left(\frac{\beta\epsilon}{2}\right) \cosh\left(\frac{\beta\epsilon}{2}\right)}{2 \cosh^2\left(\frac{\beta\epsilon}{2}\right)} \\&= -\epsilon \tanh\left(\frac{\beta\epsilon}{2}\right)\end{aligned}$$

Correct answer is (d)

23. For a particular thermodynamics system the entropy S is related to the internal energy U and volume V by

$$S = c U^{3/4} V^{1/4}$$

where c is a constant. The Gibbs potential $G = U - TS + pV$ for this system is

- (a) $\frac{3pU}{4T}$ (b) $\frac{cU}{3}$ (c) zero (d) $\frac{US}{4V}$

Soln. Entropy of the system,

$$S = c U^{3/4} V^{1/4}$$

From combined form of first and second law of thermodynamics,

$$TdS = dU + pdV \Rightarrow \frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_V, \frac{p}{T} = + \left(\frac{\partial S}{\partial V}\right)_U$$

$$\frac{1}{T} = \frac{3}{4} c U^{-1/4} V^{-1/4} = \frac{3}{4} \frac{c U^{3/4} V^{1/4}}{U^{1/4} U^{3/4}} = \frac{3S}{4U}$$

$$T = \frac{4U}{3S}$$

$$\frac{P}{T} = + \left(\frac{\partial S}{\partial V} \right)_U = + \frac{c}{4} U^{3/4} V^{-3/4} = + \frac{c}{4} \frac{U^{3/4} V^{1/4}}{V^{1/4} V^{3/4}}$$

$$P = + \frac{c}{4} \frac{U^{3/4} V^{1/4}}{V^{1/4} V^{3/4}} \times \frac{4}{3} \frac{U}{c U^{3/4} V^{1/4}} = + \frac{1}{3} \frac{U}{V}$$

$$\text{Gibbs potential, } G = U - TS + pV = U - \frac{4}{3}U + \frac{1}{3}U = 0$$

Correct answer is (c)

24. An op-amp based voltage follower

- (a) is useful for converting a low impedance source into a high impedance source
- (b) is useful for converting a high impedance source into a low impedance source
- (c) has infinitely high closed loop output impedance
- (d) has infinitely high closed loop gain

Soln. Is useful for converting a high impedance source into a low impedance source. At the input op-amp impedance is high and at output impedance is low.

Correct answer is (b)

25. A particle of mass m in three dimensions is in the potential

$$V(r) = \begin{cases} 0 & r < a \\ \infty & r \geq a \end{cases}$$

Its ground state energy is

- (a) $\frac{\pi^2 \hbar^2}{2ma^2}$
- (b) $\frac{\pi^2 \hbar^2}{ma^2}$
- (c) $\frac{3\pi^2 \hbar^2}{2ma^2}$
- (d) $\frac{9\pi^2 \hbar^2}{2ma^2}$

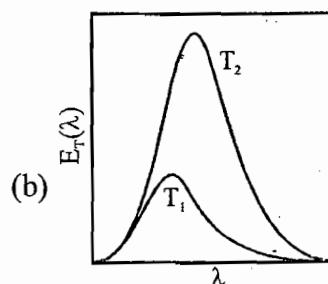
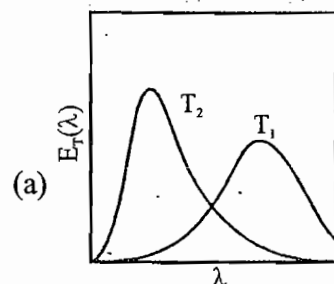
Soln. The energy eigenvalues of the given infinite spherical square well potential is

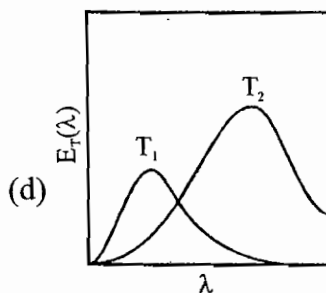
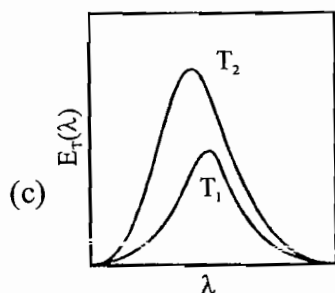
$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \quad \text{for } \ell = 0 \quad (n = 1, 2, 3, \dots)$$

The ground state energy will be $\frac{\pi^2 \hbar^2}{2ma^2}$

Correct answer is (a)

26. Which of the graphs below gives the correct qualitative behavior of the energy density $E_T(\lambda)$ of blackbody radiation of wavelength λ at two temperatures T_1 and T_2 ($T_1 < T_2$)?





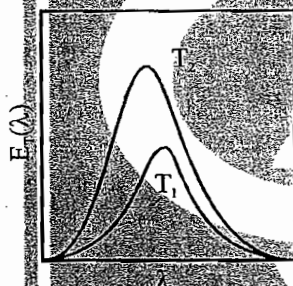
Soln. As a consequence of Wien's displacement law, we can relate how the spectrum of black body radiation depends on the temperature of the body and corresponding mathematical equation is,

$$\lambda_m T = \text{constant}$$

where, λ_m is the wavelength at which the energy density of black body radiation, is maximum at temperature T i.e. for two temperatures T_1 and T_2 .

$$\frac{(\lambda_m)_1}{(\lambda_m)_2} = \frac{T_2}{T_1}$$

Since, $T_2 > T_1 \Rightarrow (\lambda_m)_1 > (\lambda_m)_2$ and the total energy density is proportional to T^4 . Therefore,



Correct answer is (c)

27. Given that $\hat{p}_r = -i\hbar \left(\frac{\partial}{\partial r} + \frac{1}{r} \right)$, the uncertainty Δp_r in the ground state

$$\psi_0(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

of the hydrogen atom is

- (a) $\frac{\hbar}{a_0}$ (b) $\frac{\sqrt{2}\hbar}{a_0}$ (c) $\frac{\hbar}{2a_0}$ (d) $\frac{2\hbar}{a_0}$

Soln. $\hat{p}_r = -i\hbar \left(\frac{\partial}{\partial r} + \frac{1}{r} \right)$

$$\hat{p}_r^2 = (-i\hbar)^2 \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) = -\hbar^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right)$$

Given : Ground state wave function $\psi_0 = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$

$$\langle \hat{p}_r \rangle = \langle \psi_0 | \hat{p}_r | \psi_0 \rangle = 0$$

$$\langle \hat{p}_r^2 \rangle = \int_0^\infty \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \left(-\hbar^2 \right) \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \left(\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \right) 4\pi r^2 dr$$

$$= \int_0^{\infty} \frac{1}{\pi a_0^3} (-\hbar^2) e^{-r/a_0} \left[\frac{1}{a_0^2} e^{-r/a_0} + \frac{2}{r} \left(-\frac{1}{a_0} \right) e^{-r/a_0} \right] 4\pi r^2 dr$$

$$= -\frac{\hbar^2 4\pi}{\pi a_0^3} \left[\frac{1}{a_0^2} \int_0^{\infty} e^{-2r/a_0} r^2 dr - \frac{2}{a_0} \int_0^{\infty} e^{-2r/a_0} r dr \right]$$

Using the integration : $\int_0^{\infty} e^{-ar} r^n dr = \frac{n!}{a^{n+1}}$

$$= -\frac{4\hbar^2}{a_0^3} \left[\frac{1}{a_0^2} \frac{2!}{(2/a_0)^3} - \frac{2}{a_0} \frac{1!}{(2/a_0)^2} \right]$$

$$= -\frac{4\hbar^2}{a_0^3} \left[\frac{a_0}{4} - \frac{a_0}{2} \right] = \frac{\hbar^2}{a_0^2}$$

Momentum spread = $\Delta p_r = \sqrt{\langle p_r^2 \rangle - \langle p_r \rangle^2} = \frac{\hbar}{a_0}$

Correct answer is (a)

28. An RC network produces a phase shift of 30° . How many such RC networks should be cascaded together and connected to a Common Emitter amplifier so that the final circuit behaves as an oscillator?

(a) 6 (b) 12 (c) 9 (d) 3

Soln. For an RC-phase shift oscillator 3 stages provide 180° . Hence, each stage provides a phase shift of 60° .

If each stage provides phase shift of 30° then 6 RC stages are used.

Correct answer is (a)

29. The free energy F of a system depends on a thermodynamics variable ψ as

$$F = -a\psi^2 + b\psi^6$$

with $a, b > 0$. The value of ψ , when the system is in thermodynamic equilibrium, is

(a) zero (b) $\pm(a/6b)^{1/4}$ (c) $\pm(a/3b)^{1/4}$ (d) $\pm(a/b)^{1/4}$

Soln. Free energy of system,

$$F = -a\psi^2 + b\psi^6, \quad a, b > 0$$

In equilibrium,

$$\frac{\partial F}{\partial \psi} = 0 \quad \& \quad \frac{\partial^2 F}{\partial \psi^2} > 0$$

$$-2a\psi = 6b\psi^5 = 0$$

$$\psi(-2a + 6b\psi^4) = 0$$

$$\psi = 0$$

$$-2a + 6b\psi^4 = 0$$

$$\psi^4 = \frac{2a}{6b}$$

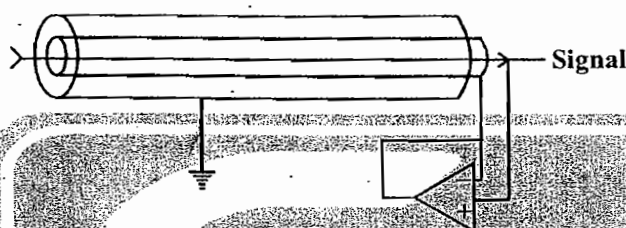
$$\psi = \pm \left(\frac{a}{3b} \right)^{1/4}$$

$$\text{At } \psi = 0, \frac{d^2 F}{d\psi^2} < 0$$

$$\text{But at } \psi = \pm \left(\frac{a}{3b} \right)^{1/4}, \frac{\partial^2 F}{\partial \psi^2} > 0$$

Correct answer is (c)

30. The inner shield of a triaxial conductor is driven by an (ideal) op-amp follower circuit as shown. The effective capacitance between the signal-carrying conductor and ground is



- (a) unaffected (b) doubled (c) halved (d) made zero

Soln. For a voltage follow ideal op-amp. The effective capacitance between signal carrying conductor and grand can be made either zero or unaffected (a) or (d).

Correct answer is (d)

31. Consider a system of two non-interacting identical fermions each of mass m in an infinite square well potential of width a . (Take the potential inside the well to be zero and ignore spin). The composite wavefunction for the system with total energy

$$E = \frac{5\pi^2 \hbar^2}{2ma^2}$$

is

- (a) $\frac{2}{a} \left[\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \right]$
 (b) $\frac{2}{a} \left[\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \right]$
 (c) $\frac{2}{a} \left[\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{3\pi x_2}{2a}\right) - \sin\left(\frac{3\pi x_1}{2a}\right) \sin\left(\frac{\pi x_2}{a}\right) \right]$
 (d) $\frac{2}{a} \left[\sin\left(\frac{\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right) - \sin\left(\frac{\pi x_2}{a}\right) \sin\left(\frac{\pi x_1}{a}\right) \right]$

Soln. A system of two non-interacting identical fermions have anti-symmetric wave function. If we ignore the spin of the fermions, then space part of the wave function will be anti-symmetric.

$$\text{Energy of the system} = \frac{\pi^2 \hbar^2}{2ma^2} (n_1^2 + n_2^2)$$

$$\text{Given : } \frac{\pi^2 \hbar^2}{2ma^2} (n_1^2 + n_2^2) = \frac{5\pi^2 \hbar^2}{2ma^2} \Rightarrow n_1 = 1, n_2 = 2$$

The space part wavefunction of the system

$$= \frac{1}{\sqrt{2}} [\psi_{n_1}(x_1) \psi_{n_2}(x_2) + \psi_{n_2}(x_1) \psi_{n_1}(x_2)]$$

$$= \frac{1}{\sqrt{2}} \left[\frac{2}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) - \frac{2}{a} \sin\left(\frac{2\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \right]$$

Correct answer is (a)

32. A particle of mass m in the potential $V(x, y) = \frac{1}{2} m \omega^2 (4x^2 + y^2)$, is in an eigenstate of energy

$E = \frac{5}{2} \hbar \omega$. The corresponding un-normalized eigenfunction is

- (a) $y \exp\left[-\frac{m\omega}{2\hbar}(2x^2 + y^2)\right]$ (b) $x \exp\left[-\frac{m\omega}{2\hbar}(2x^2 + y^2)\right]$
 (c) $y \exp\left[-\frac{m\omega}{2\hbar}(x^2 + y^2)\right]$ (d) $xy \exp\left[-\frac{m\omega}{2\hbar}(x^2 + y^2)\right]$

Soln. $V(x, y) = \frac{1}{2} m \omega^2 (4x^2 + y^2) = \frac{1}{2} m (2\omega)^2 x^2 + \frac{1}{2} m (\omega)^2 y^2$

It is a 2-D anisotropic harmonic oscillator having $\omega_x = 2\omega$ and $\omega_y = \omega$

Energy eigenvalue = $E_{n_x n_y} = \left(n_x + \frac{1}{2}\right) \hbar (2\omega) + \left(n_y + \frac{1}{2}\right) \hbar (\omega) = \left(n + \frac{3}{2}\right) \hbar \omega$ (where, $n = 2n_x + n_y$)

Given : $E_{n_x n_y} = \frac{5}{2} \hbar \omega \Rightarrow n = 0, n = 1$

The corresponding wavefunction will be

$\psi_{n_x n_y}(x, y) = A_x e^{-\alpha_x^2 x^2 / 2} H_{n_x}(\alpha_x x) A_y e^{-\alpha_y^2 y^2 / 2} H_{n_y}(\alpha_y y)$ where, $\alpha_x^2 = \frac{m(2\omega)}{\hbar}$, $\alpha_y^2 = \frac{m(\omega)}{\hbar}$

$\psi_{01}(x, y) = A_x A_y H_0(\alpha_x x) H_1(\alpha_y y) \exp\left[-\frac{1}{2}(\alpha_x^2 x^2 + \alpha_y^2 y^2)\right]$

$\Rightarrow \psi_{01}(x, y) = A_x A_y (1)(2\alpha_y y) \exp\left[-\frac{m\omega}{2\hbar}(2x^2 + y^2)\right]$

Correct answer is (a)

33. A particle of mass m and coordinate q has the Lagrangian

$$L = \frac{1}{2} m \dot{q}^2 - \frac{\lambda}{2} q \dot{q}^2$$

where λ is a constant. The Hamiltonian for the system is given by

- (a) $\frac{p^2}{2m} + \frac{\lambda q p^2}{2m^2}$ (b) $\frac{p^2}{2(m - \lambda q)}$ (c) $\frac{p^2}{2m} + \frac{\lambda q p^2}{2(m - \lambda q)^2}$ (d) $\frac{pq}{2}$

Soln. $L = \frac{1}{2} m \dot{q}^2 - \frac{\lambda}{2} q \dot{q}^2$, $p = \frac{\partial L}{\partial \dot{q}} = m \dot{q} - \lambda q \dot{q}$

$\therefore \dot{q} = \frac{p}{m - \lambda q}$

$H = p \dot{q} - L$

$$\begin{aligned}
 &= p\dot{q} - \frac{1}{2}m\dot{q}^2 + \frac{\lambda}{2}q\dot{q}^2 \\
 &= \frac{p^2}{m-\lambda q} - \frac{1}{2}m\frac{p^2}{(m-\lambda q)^2} + \frac{\lambda q}{2}\frac{p^2}{(m-\lambda q)^2} \\
 &= \frac{p^2}{m-\lambda q} - \frac{1}{2}(m-\lambda q)\frac{p^2}{(m-\lambda q)^2} \\
 &= \frac{p^2}{m-\lambda q} - \frac{p^2}{2(m-\lambda q)} = \frac{p^2}{2(m-\lambda q)}
 \end{aligned}$$

Correct answer is (b)

34. If $\vec{A} = yz\hat{i} + xz\hat{j} + xy\hat{k}$ and C is the circle of unit radius in the plane defined by $z = 1$, with the centre on the z -axis, then the value of the integral $\oint_C \vec{A} \cdot d\vec{\ell}$ is

- (a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{4}$ (d) 0

Soln. $\vec{A} = yz\hat{i} + xz\hat{j} + xy\hat{k} \Rightarrow \nabla \times \vec{A} = 0$ i.e. field is conservative.

Since, line integral of a conservative field along a closed path is zero, then $\oint_C \vec{A} \cdot d\vec{\ell} = 0$

Correct answer is (d)

35. Given,

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-1/2}, \text{ for } |t| < 1, \text{ the value of } P_5(-1) \text{ is}$$

- (a) 0.26 (b) 1 (c) 0.5 (d) 1

Soln. $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$

Putting $x = -1$ in the above equation, we get

$$(1 + 2t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1) t^n \Rightarrow (1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 - \dots = \sum_{n=0}^{\infty} P_n(-1) t^n$$

Comparing the coefficient t^5 on the both sides, we get $P_5(-1) = -1$

Correct answer is (d)

36. A charged particle is at a distance d from an infinite conducting plane maintained at zero potential. When released from rest, the particle reaches a speed u at a distance $d/2$ from the plane. At what distance from the plane will the particle reach the speed $2u$?

- (a) $d/6$ (b) $d/3$ (c) $d/4$ (d) $d/5$

37. Consider the matrix

$$M = \begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$$

The eigenvalues of M are

- (a) $-5, -2, 7$ (b) $-7, 0, 7$ (c) $-4i, 2i, 2i$ (d) $2, 3, 6$

Soln. Sum of the eigenvalues of the matrix $M = \text{trace of the matrix } M = 0$

Product of the eigenvalues of the matrix $M = \text{determinant of the matrix } M = 0$

Only $-7, 0, 7$ satisfies both the relation.

Correct answer is (b)

38. Consider the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$

with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. The solution $x(t)$ attains its maximum value when 't' is

- (a) $1/2$ (b) 1 (c) 2 (d) ∞

Soln. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$

Assume the trial solution $x = ce^{mt}$
 $\Rightarrow (m^2 + 2m + 1)ce^{mt} = 0 \Rightarrow (m^2 + 2m + 1) = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

General solution $= x(t) = (c_1 + c_2 t)e^{-t}$

Putting $x(0) = 0 \Rightarrow c_1 = 0$. So, $x(t) = c_2 t e^{-t}$

Since, $x(t) = c_2 t e^{-t} \Rightarrow \frac{dx}{dt} = c_2 e^{-t} - c_2 t e^{-t}$

Putting $\dot{x}(0) = 1 \Rightarrow c_2 = 1$. So, $x(t) = t e^{-t}$

Checking for maxima and minima

$$\therefore x(t) = t e^{-t} \Rightarrow \frac{dx}{dt} = e^{-t} - t e^{-t} \Rightarrow \frac{d^2x}{dt^2} = -e^{-t} - 2e^{-t}$$

For maxima or minima, $\frac{dx}{dt} = 0 \Rightarrow t = 1$

$$\text{At } t = 1, \frac{d^2x}{dt^2} = -e^{-1} = -ve$$

Therefore, $x(t)$ attains its maximum value at $t = 1$

Correct answer is (b)

39. A light source is switched on and off at a constant frequency f . An observer moving with a velocity u with respect to the light source will observe the frequency of the switching to be

- (a) $f \left(1 - \frac{u^2}{c^2}\right)^{-1}$ (b) $f \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$ (c) $f \left(1 - \frac{u^2}{c^2}\right)$ (d) $f \left(1 - \frac{u^2}{c^2}\right)^{1/2}$

Soln. From time dilation formula

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \therefore \frac{1}{T} = \frac{1}{T_0} \sqrt{1 - \frac{v^2}{c^2}} \Rightarrow f = f_0 \sqrt{1 - \frac{v^2}{c^2}}$$

Correct answer is (d)

40. If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral

$$\oint_C \frac{dz}{\sin^2 z}$$

is

- (a) ∞ (b) $2\pi i$ (c) 0 (d) πi

Soln. $f(z) = \frac{1}{\sin^2 z}$ has poles of order 2 at $z = n\pi$ ($n = 0, 1, 2, \dots$)

Only $z = 0$ will be within the circle $|z| = \frac{1}{2}$

The given function, $f(z) = \frac{1}{\sin^2 z}$ can be expanded into Laurent series about $z = 0$ as following:

$$\begin{aligned} f(z) &= \frac{1}{\sin^2 z} = \frac{1}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)^2} = \frac{1}{z^2 \left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right]^2} \\ &= \frac{1}{z^2} \left[1 - \left\{\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right\}\right]^{-2} \\ &= \frac{1}{z^2} \left[1 + 2\left\{\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right\} + 3\left\{\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right\}^2 + \dots\right] \end{aligned}$$

Residue of $f(z)$ at $(z = 0)$ = coefficient of $\left(\frac{1}{z}\right)$ in the Laurent series expansion of $f(z)$ about $(z = 0)$.

Therefore, the value of the integral = 0

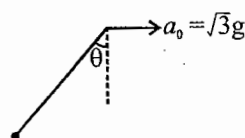
Correct answer is (c).

41. The time period of a simple pendulum under the influence of the acceleration due to gravity g is T . The bob is subjected to an additional acceleration of magnitude $\sqrt{3}g$ in the horizontal direction. Assuming small oscillations, the mean position and time period of oscillation, respectively, of the bob will be

- (a) 0° to the vertical and $\sqrt{3}T$ (b) 30° to the vertical and $T/2$
(c) 60° to the vertical and $T/\sqrt{2}$ (d) 0° to the vertical and $T/\sqrt{3}$

Soln. $T = 2\pi \sqrt{\frac{l}{g}}$

$$T' = 2\pi \sqrt{\frac{l}{\sqrt{g^2 + a_0^2}}} = 2\pi \sqrt{\frac{l}{2g}} = \frac{T}{\sqrt{2}}$$



$$\tan \theta = \frac{a_0}{g} = \frac{\sqrt{3}g}{g} = \sqrt{3}$$

$$\therefore \theta = 60^\circ$$

Correct answer is (c)

42. Consider an electromagnetic wave at the interface between two homogeneous dielectric media of the dielectric constants ϵ_1 and ϵ_2 . Assuming $\epsilon_2 > \epsilon_1$ and non charges on the surface, the electric field vector $\vec{E}$ and the displacement vector $\vec{D}$ in the two media satisfy the following inequalities
- (a) $|\vec{E}_2| > |\vec{E}_1|$ and $|\vec{D}_2| > |\vec{D}_1|$ (b) $|\vec{E}_2| < |\vec{E}_1|$ and $|\vec{D}_2| < |\vec{D}_1|$
- (c) $|\vec{E}_2| < |\vec{E}_1|$ and $|\vec{D}_2| > |\vec{D}_1|$ (d) $|\vec{E}_2| > |\vec{E}_1|$ and $|\vec{D}_2| < |\vec{D}_1|$

Soln. $E_1 = \sqrt{(E_1^\parallel)^2 + (E_1^\perp)^2}$, $E_2 = \sqrt{(E_2^\parallel)^2 + (E_2^\perp)^2}$

$$D_1 = \sqrt{(D_1^\parallel)^2 + (D_1^\perp)^2}, D_2 = \sqrt{(D_2^\parallel)^2 + (D_2^\perp)^2}$$

Boundary conditions

$$E_1^\parallel = E_2^\parallel$$

$$D_1^\perp = D_2^\perp$$

From (i) $\frac{D_1^\parallel}{\epsilon_1} = \frac{D_2^\parallel}{\epsilon_2}$, $\frac{D_1^\parallel}{D_2^\parallel} = \frac{\epsilon_1}{\epsilon_2}$

$$\Rightarrow D_1^\parallel < D_2^\parallel$$

from (ii) $\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp$

$$\frac{E_1^\perp}{E_2^\perp} = \frac{\epsilon_2}{\epsilon_1}$$

$$\Rightarrow E_1^\perp > E_2^\perp$$

Using (i) and (iv) we get $E_1 > E_2$, using equation (ii) and (iii) we get $D_2 > D_1$

Correct answer is (c)

43. If the electrostatic potential in spherical polar coordinates is

$$\phi(r) = \phi_0 e^{-r/r_0}$$

where ϕ_0 and r_0 are constants, then the charge density at a distance $r = r_0$ will be

(a) $\frac{\epsilon_0 \phi_0}{er_0^2}$

(b) $\frac{e\epsilon_0 \phi_0}{2r_0^2}$

(c) $-\frac{\epsilon_0 \phi_0}{er_0^2}$

(d) $-\frac{2e\epsilon_0 \phi_0}{r_0^2}$

Soln. Poisson's equation, $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = -\frac{\rho}{\epsilon_0}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \phi_0 e^{-r/r_0} \right) = -\frac{\rho}{\epsilon_0}$$

$$\frac{-\phi_0}{r_0 r^2} \frac{\partial}{\partial r} \left(r^2 e^{-r/r_0} \right) = -\frac{\rho}{\epsilon_0}$$

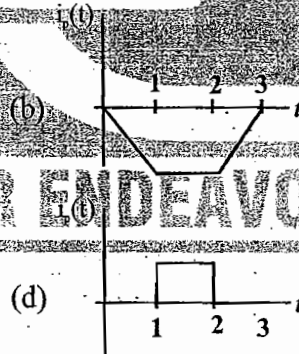
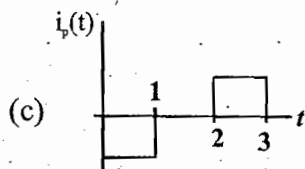
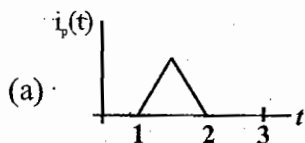
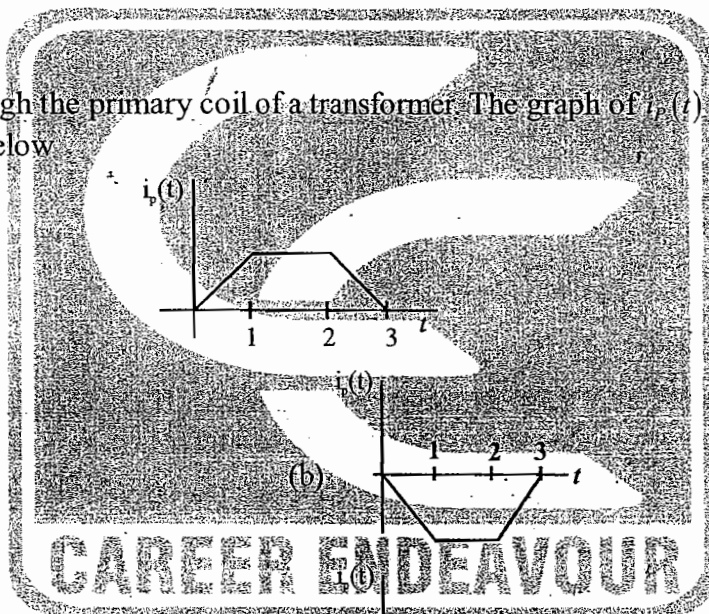
$$\frac{\phi_0}{r_0 r^2} \left[2r e^{-r/r_0} - \frac{r^2}{r_0} e^{-r/r_0} \right] = \frac{\rho}{\epsilon_0}$$

$$\text{at } r = r_0 \quad \frac{\phi_0}{r_0^3} [2r_0 e^{-1} - r_0 e^{-1}] = \frac{\rho}{\epsilon_0}, \quad \frac{\phi_0 e^{-1}}{r_0^2} = \frac{\rho}{\epsilon_0}$$

$$\rho = \frac{\phi_0 \epsilon_0}{r_0^2 e}$$

Correct answer is (a)

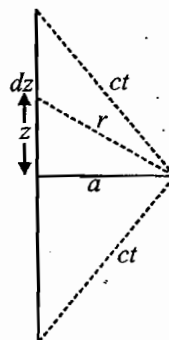
44. A current i_p flows through the primary coil of a transformer. The graph of $i_p(t)$ as a function of time 't' is shown in figure below



Soln.
$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I \left(t - \frac{r}{c} \right) d\vec{l}}{r}$$

$$= \frac{\mu_0}{4\pi} \int_{-\sqrt{c^2 t^2 - a^2}}^{\sqrt{c^2 t^2 - a^2}} \frac{k \left(t - \frac{\sqrt{z^2 + a^2}}{c} \right) dz \hat{z}}{\sqrt{z^2 + a^2}}$$

$$= \frac{\hat{z} \mu_0}{4\pi c} k \int_{-\sqrt{c^2 t^2 - a^2}}^{\sqrt{c^2 t^2 - a^2}} dz \frac{ct - \sqrt{a^2 + z^2}}{(a^2 + z^2)^{1/2}}$$



Correct answer is (a)

45. A time-dependent current $\vec{I}(t) = Kt\hat{z}$ (where K is a constant) is switched on at $t = 0$ in an infinite current-carrying wire. The magnetic vector potential at a perpendicular distance 'a' from the wire is given (for time $t > a/c$) by

(a) $\hat{z} \frac{\mu_0 K}{4\pi c} \int_{-\sqrt{c^2 t^2 - a^2}}^{\sqrt{c^2 t^2 - a^2}} dz \frac{ct - \sqrt{a^2 + z^2}}{(a^2 + z^2)^{1/2}}$

(b) $\hat{z} \frac{\mu_0 K}{4\pi} \int_{-ct}^{ct} dz \frac{t}{(a^2 + z^2)^{1/2}}$

(c) $\hat{z} \frac{\mu_0 K}{4\pi c} \int_{-ct}^{ct} dz \frac{ct - \sqrt{a^2 + z^2}}{(a^2 + z^2)^{1/2}}$

(d) $\hat{z} \frac{\mu_0 K}{4\pi} \int_{-\sqrt{c^2 t^2 - a^2}}^{\sqrt{c^2 t^2 - a^2}} dz \frac{t}{(a^2 + z^2)^{1/2}}$

Soln. Correct answer is (c)

PART-C

46. The pressure of a non-relativistic free Fermi gas in three-dimensions depends, at $T = 0$, on the density of fermions n as

(a) $n^{5/3}$

(b) $n^{1/3}$

(c) $n^{2/3}$

(d) $n^{4/3}$

Soln. $E_F = \frac{h^2}{2m} \left(\frac{3}{4\pi g_s} n \right)^{2/3}$

$$P_0 = \frac{2}{5} n E_F \propto n^{5/3}$$

Correct answer is (a)

47. A double slit interference experiment uses a laser emitting light of two adjacent frequencies ν_1 and ν_2 ($\nu_1 < \nu_2$). The minimum path difference between the interfering beams for which the interference pattern disappears is

(a) $\frac{c}{\nu_2 + \nu_1}$

(b) $\frac{c}{\nu_2 - \nu_1}$

(c) $2(\nu_2 - \nu_1)$

(d) $2(\nu_2 + \nu_1)$

Soln. Correct answer is (d)

48. The recently-discovered Higgs boson at the LHC experiment has a decay mode into a photon and a Z boson. If the rest masses of the Higgs and Z boson are $125 \text{ GeV}/c^2$ and $90 \text{ GeV}/c^2$ respectively, and the decaying Higgs particle is at rest, the energy of the photon will approximately be

(a) $35\sqrt{3} \text{ GeV}$

(b) 35 GeV

(c) 30 GeV

(d) 15 GeV

Soln. The momentum of photon will be same (in magnitude) as Z-boson. If the energy of photon is E , then momentum

$$p = \frac{E}{c}$$

$$\text{Now, energy of Z-boson} = \sqrt{p^2 c^2 + m_z^2 c^4}$$

where, m_z = rest mass of Z-boson.

Now, initial energy = final energy.

Rest energy of Higgs boson = energy of Z-boson + energy of photon.

$$\begin{aligned} \Rightarrow \sqrt{E^2 + 90^2} + E &= 125 \text{ MeV} \\ \Rightarrow E^2 + 90^2 &= (125 - E)^2 \\ \Rightarrow E^2 + 90^2 &= (125)^2 + E^2 - 250E \\ \Rightarrow 250E &= (125 + 90)(125 - 90) \\ \Rightarrow 250E &= 215 \times 35 \\ \Rightarrow E &= 30.1 \text{ GeV} \end{aligned}$$

Correct answer is (c)

49. A permanently deformed even-even nucleus with $J^P = 2^+$ has rotational energy 93 keV. The energy of the next excited state is
 (a) 372 keV (b) 310 keV (c) 273 keV (d) 186 keV

Soln. The energy $= \frac{\hbar^2}{2I} J(J+1)$, where, I is the moment of inertia.

So, for $J = 2$

$$\frac{\hbar^2}{2I} \times 2 \times 3 = 93 \Rightarrow \frac{\hbar^2}{I} = 31$$

The next excited state is $J^P = 4^+$

$$\text{So, energy} = \frac{\hbar^2}{2I} \times 4 \times 5 = \frac{\hbar^2}{I} \times 10 = 310 \text{ keV}$$

Correct answer is (b)

50. How much does the total angular momentum quantum number J change in the transition of $\text{Cr}(3d^6)$ atom as it ionizes to $\text{Cr}^{2+}(3d^4)$?
 (a) increases by 2 (b) decreases by 2 (c) decreases by 4 (d) does not change

Soln. $\text{Cr}(3d^6) \Rightarrow$

$m_l = 2$	1	0	-1	-2
$\uparrow\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$

For ground state term, $\ell = \sum m_l = 2$

$$s = \sum m_s = 2$$

$$j = 4, 3, 2, 1, 0$$

Since, $3d^6$ is greater than half filled, then ground state will correspond to maximum value of j .

Ground state term $= {}^5D_4$

$\text{Cr}(3d^4) \Rightarrow$

$m_l = 2$	1	0	-1	-2
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	

For ground state term, $\ell = \sum m_l = 2$

$$s = \sum m_s = 2$$

$$j = 4, 3, 2, 1, 0$$

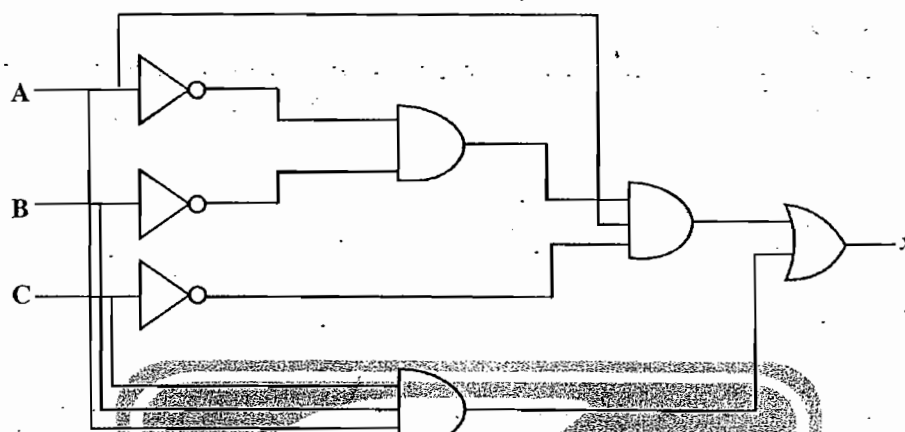
Since, $3d^4$ is less than half filled, the ground state will correspond to minimum value of j .

Ground state term = $^5D_0$

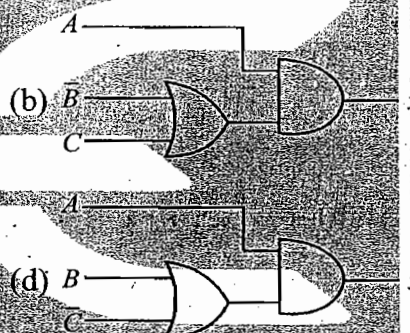
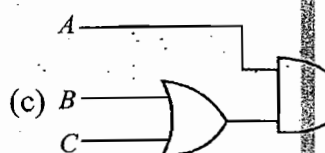
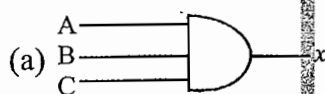
Therefore, total angular momentum quantum number decreases by 4.

Correct answer is (c)

51. For the logic circuit shown in the figure below



a simplified equivalent circuit is



Soln. The output expression $x = \overline{ABC}$

Correct answer is (a)

52. A spectral line due to a transition from an electronic state p to an s state splits into three Zeeman lines in the presence of a strong magnetic field. At intermediate field strengths the number of spectral lines is

(a) 10

(b) 3

(c) 6

(d) 9

Soln. For p state, $\ell = 1, s = \frac{1}{2} \Rightarrow j = \frac{3}{2}, \frac{1}{2}$

Corresponding values of $m_j = \pm \frac{3}{2}, \pm \frac{1}{2}$ and $m_j = \pm \frac{1}{2}$

For s state, $\ell = 0, s = \frac{1}{2} \Rightarrow j = \frac{1}{2}$

Corresponding values of $m_j = \pm \frac{1}{2}$

There are 12 different combinations of m_j between final and initial level. But $m_j = \frac{3}{2} \rightarrow m_j = -\frac{1}{2}$

and $m_j = -\frac{3}{2} \rightarrow m_j = \frac{1}{2}$ is forbidden according to selection rule : $\Delta j = 0, \pm 1$.

Number of spectral lines = $12 - 2 = 10$

Correct answer is (a)

53. A particle in the infinite square well

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{otherwise} \end{cases}$$

is prepared in a state with the wavefunction

$$\psi(x) = \begin{cases} A \sin^3\left(\frac{\pi x}{a}\right) & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

The expectation value of the energy of the particle is

(a) $\frac{5\hbar^2\pi^2}{2ma^2}$ (b) $\frac{9\hbar^2\pi^2}{2ma^2}$ (c) $\frac{9\hbar^2\pi^2}{10ma^2}$ (d) $\frac{\hbar^2\pi^2}{2ma^2}$

Soln. $\psi(x) = A \sin^3\left(\frac{\pi x}{a}\right) = \frac{3A}{4} \sin\left(\frac{\pi x}{a}\right) - \frac{A}{4} \sin\left(\frac{3\pi x}{a}\right)$

$$\Rightarrow \psi(x) = \frac{3A}{4} \sqrt{\frac{a}{2}} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \right] - \frac{A}{4} \sqrt{\frac{a}{2}} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right) \right]$$

Since, $\psi(x)$ is normalized, then

$$\left[\frac{3A}{4} \sqrt{\frac{a}{2}} \right]^2 + \left[\frac{A}{4} \sqrt{\frac{a}{2}} \right]^2 = 1 \Rightarrow \frac{9A^2}{16} \frac{a}{2} + \frac{A^2}{16} \frac{a}{2} = 1 \Rightarrow A = \sqrt{\frac{16}{5a}} = \frac{4}{\sqrt{5a}}$$

Therefore, $\psi(x) = \frac{3}{\sqrt{10}} \psi_1 - \frac{1}{\sqrt{10}} \psi_3$

Expectation value of energy = $\langle E \rangle = P_1 E_1 + P_3 E_3 = \frac{9}{10} \frac{\pi^2 \hbar^2}{2ma^2} + \frac{1}{10} \frac{9\pi^2 \hbar^2}{2ma^2} = \frac{9\pi^2 \hbar^2}{10ma^2}$

Correct answer is (c).

54. The average local internal magnetic field acting on an Ising spin is $H_{\text{int}} = \alpha M$, where M is the magnetization and α is a positive constant. At a temperature T sufficiently close to (and above) the critical temperature T_c , the magnetic susceptibility at zero external field is proportional to (k_B is the Boltzmann constant)

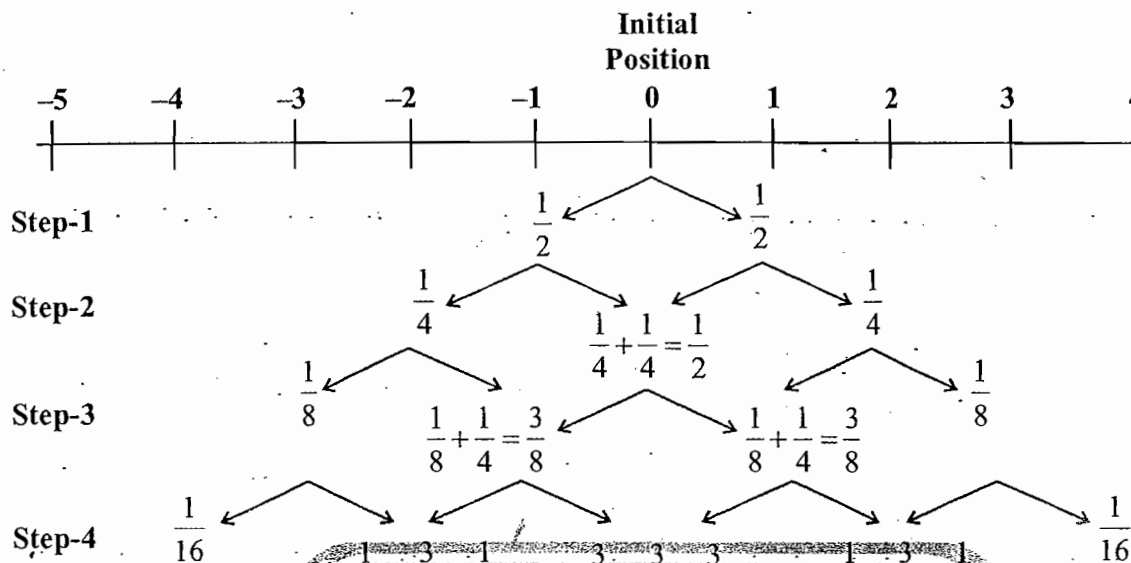
(a) $k_B T - \alpha$ (b) $(k_B T + \alpha)^{-1}$ (c) $(k_B T - \alpha)^{-1}$ (d) $\tanh(k_B T + \alpha)$

Soln. Correct answer is (c)

55. In one dimension, a random walker takes a step with equal probability to the left or right. What is the probability that the walker returns to the starting point after 4 steps?

(a) $3/8$ (b) $5/16$ (c) $1/4$ (d) $1/16$

Soln. First Method:



The probability that the walker will return to the starting point after 4 steps = $3/8$

Second Method:

In one dimension the probability of taking r steps right out of N steps

$$= {}^N C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{N-r}$$

$$\text{So, } P = {}^4 C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = \frac{4 \times 3 \times 2!}{2! 2!} \frac{1}{2^4} = \frac{6}{16} = \frac{3}{8}$$

Correct answer is (a)

56. Consider an electron in a b.c.c. lattice with lattice constant a . A single particle wavefunction that satisfies the Bloch theorem will have the form $f(\vec{r})\exp(i\vec{k}\cdot\vec{r})$, with $f(\vec{r})$ being

- (a) $1 + \cos\left[\frac{2\pi}{a}(x+y-z)\right] + \cos\left[\frac{2\pi}{a}(-x+y+z)\right] + \cos\left[\frac{2\pi}{a}(x-y+z)\right]$
- (b) $1 + \cos\left[\frac{2\pi}{a}(x+y)\right] + \cos\left[\frac{2\pi}{a}(y+z)\right] + \cos\left[\frac{2\pi}{a}(z+x)\right]$
- (c) $1 + \cos\left[\frac{\pi}{a}(x+y)\right] + \cos\left[\frac{\pi}{a}(y+z)\right] + \cos\left[\frac{\pi}{a}(z+x)\right]$
- (d) $1 + \cos\left[\frac{\pi}{a}(x+y-z)\right] + \cos\left[\frac{\pi}{a}(-x+y+z)\right] + \cos\left[\frac{\pi}{a}(x-y+z)\right]$

Soln. The reciprocal of BCC will be FCC i.e. block theorem satisfy the wave equation.

$$f(r) = 1 + \cos\left[\frac{2\pi}{a}(x+y)\right] + \cos\left[\frac{2\pi}{a}(y+z)\right] + \cos\left[\frac{2\pi}{a}(z+x)\right]$$

Correct answer is (b)

Note : $E_n(K+G) = E_n(k)$. This indicates that $E_n k$ is periodic with an period equal to the reciprocal lattice vector $\left(G = \frac{2\pi}{a}\right)$.

57. The dispersion relation for electrons in an f.c.c. crystal is given, in the tight binding approximation by

$$\varepsilon(k) = -4\varepsilon_0 \left[\cos \frac{k_x a}{2} \cos \frac{k_y a}{2} + \cos \frac{k_y a}{2} \cos \frac{k_z a}{2} + \cos \frac{k_z a}{2} \cos \frac{k_x a}{2} \right]$$

where 'a' is the lattice constant and ε_0 is a constant with the dimension of energy. The x-component of the velocity of the electrons at $\left(\frac{\pi}{a}, 0, 0\right)$ is

- (a) $-\frac{2\varepsilon_0 a}{\hbar}$ (b) $\frac{2\varepsilon_0 a}{\hbar}$ (c) $-\frac{4\varepsilon_0 a}{\hbar}$ (d) $\frac{4\varepsilon_0 a}{\hbar}$

Soln. To find the x-component of velocity

$$v = \frac{1}{\hbar} \frac{dE}{dk} = \frac{1}{\hbar} \frac{dE(k)}{dk_x}$$

$$= \frac{1}{\hbar} (-4\varepsilon_0) \left[-\sin \left(\frac{k_x a}{2} \right) \left(\frac{a}{2} \right) \cos \frac{k_y a}{2} + 0 + \left(-\sin \left(\frac{k_x a}{2} \right) \left(\frac{a}{2} \right) \cos \left(\frac{k_x a}{2} \right) \right] \right]$$

$$= 0 + \frac{4\varepsilon_0}{\hbar} \left[\left(\frac{a}{2} \right) \sin \left(\frac{k_x a}{2} \right) \cos \left(\frac{k_x a}{2} \right) \right]$$

Putting $\left(\frac{\pi}{a}, 0, 0\right)$, $k_x = \frac{\pi}{a}, k_y = 0, k_z = 0$

$$= \frac{4\varepsilon_0}{\hbar} \left[\left(\frac{a}{2} \right) \sin \left(\frac{\pi}{2} \right) \cos 0 + \frac{a}{2} \sin \left(\frac{\pi}{2} \right) \cos 0 \right]$$

$$= \frac{4\varepsilon_0}{\hbar} \left[\frac{a}{2} + \frac{a}{2} \right] = \frac{4\varepsilon_0 a}{\hbar}$$

Correct answer is (d)

58. The following data is obtained in an experiment that measures the viscosity η as a function of molecular weight M for a set of polymers.

M (Da)	η (kPa-s)
990	0.28 ± 0.03
5032	30 ± 2
10191	250 ± 10
19825	2000 ± 200

The relation that best describes the dependence of η on M is

- (a) $\eta \sim M^{4/9}$ (b) $\eta \sim M^{3/2}$ (c) $\eta \sim M^2$ (d) $\eta \sim M^3$

Soln. Taking the ratio of consecutive datas for η and M .

We get, $M_2 : M_1 \approx 5 \Rightarrow \eta_2 : \eta_1 \approx 100$

$M_3 : M_2 \approx 2 \Rightarrow \eta_3 : \eta_2 \approx 8$

$M_4 : M_3 \approx 2 \Rightarrow \eta_4 : \eta_3 \approx 8$

Therefore, $\eta \sim M^3$

Correct answer is (d)

59. The integral $\int_0^1 \sqrt{x} dx$ is to be evaluated up to 3 decimal places using Simpson's 3-point rule. If the interval $[0, 1]$ is divided into 4 equal parts, the correct result is

(a) 0.683 (b) 0.667 (c) 0.657 (d) 0.638

Soln. Number of intervals = 4, width of interval = $h = b-a/n = 1-0/4 = 0.25$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x) = \sqrt{x}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

According to Simpson's 3-point rule

$$\int_0^1 \sqrt{x} dx = \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{12} \left[(0+1) + 4 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) + 2 \frac{1}{\sqrt{2}} \right] = 0.657 (\text{approx})$$

Correct answer is (c)

60. In a classical model, a scalar (spin-0) meson consists of a quark and an antiquark bound by a potential

$$V(r) = ar + \frac{b}{r}$$

where $a = 200 \text{ MeV fm}^{-1}$ and $b = 100 \text{ MeV fm}$. If the masses of the quark and antiquark are negligible, the mass of the meson can be estimated as approximately

(a) $141 \text{ MeV}/c^2$ (b) $283 \text{ MeV}/c^2$ (c) $353 \text{ MeV}/c^2$ (d) $425 \text{ MeV}/c^2$

Soln. Bound state corresponds to minimum potential energy. For that

$$\frac{\partial V}{\partial r} = 0 \text{ and } \frac{\partial^2 V}{\partial r^2} = +ve$$

From $\frac{\partial V}{\partial r} = 0 \Rightarrow \frac{\partial}{\partial r} \left(ar + \frac{b}{r} \right) = 0$

$$\Rightarrow a - \frac{b}{r^2} = 0 \Rightarrow r^2 = \frac{b}{a} \Rightarrow r = \pm \sqrt{\frac{b}{a}}$$

$r = \sqrt{\frac{b}{a}}$ corresponds to minimum potential energy.

So, the potential energy of bound state $= a \times \sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}} = 2\sqrt{ab} = 2\sqrt{200 \times 100} = 200\sqrt{2} = 283 \text{ MeV}$

This is the total energy of meson because the meson spinless.

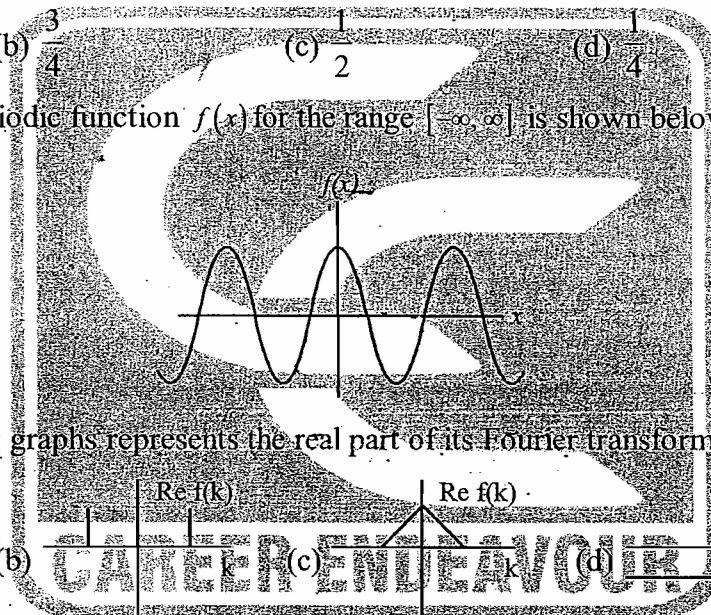
$$\text{The mass} = \frac{\text{Energy}}{c^2} = 283 \frac{\text{MeV}}{c^2}$$

Correct answer is (b)

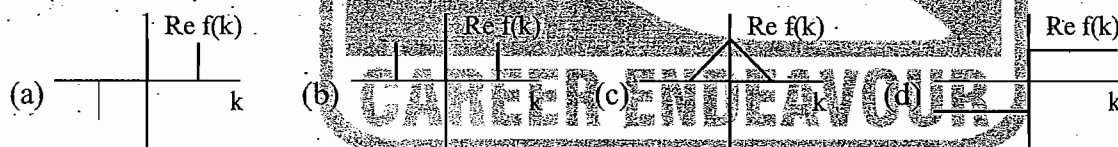
61. Let $y = \frac{1}{2}(x_1 + x_2) - \mu$, where x_1 and x_2 are independent and identically distributed Gaussian random variables of mean μ and standard deviation σ . Then $\frac{\langle y^4 \rangle}{\sigma^4}$ is

- (a) 1 (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

62. The graph of a real periodic function $f(x)$ for the range $[-\infty, \infty]$ is shown below



Which of the following graphs represents the real part of its Fourier transform?



Soln. The given graph in the question is of $f(x) = \cos \omega x$

$$F(\cos \omega x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos \omega x e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} \frac{e^{i(k+\omega)x}}{2} dx + \int_{-\infty}^{+\infty} \frac{e^{i(k-\omega)x}}{2} dx \right)$$

$$\Rightarrow F(\cos \omega x) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2} \delta(k + \omega) + \frac{1}{2} \delta(k - \omega) \right)$$

Therefore, the fourier transform of $f(x) = \cos \omega x$ consist of two delta functions of same height at $k = \omega$ and $k = -\omega$.

Correct answer is (b)

63. The matrices

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfy the commutation relations

(a) $[A, B] = B + C, [B, C] = 0, [C, A] = B + C$

(b) $[A, B] = C, [B, C] = A, [C, A] = B$

(c) $[A, B] = B, [B, C] = 0, [C, A] = A$

(d) $[A, B] = C, [B, C] = 0, [C, A] = B$

soln. The given matrix A, B, C satisfy the relations given in option (d)

Correct answer is (d)

54. The function $\Phi(x, y, z, t) = \cos(z - vt) + \text{Re}(\sin(x + iy))$ satisfies the equation

(a) $\frac{1}{v^2} \frac{\partial^2 \Phi}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi$

(b) $\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi$

(c) $\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi$

(d) $\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi$

soln. $\phi(x, y, z, t) = \cos(z - vt) + \text{Re}(\sin(x + iy))$

As $\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \cosh y$

$\Rightarrow \phi(x, y, z, t) = \cos(z - vt) + \sin x \cosh y$

We get, $\frac{\partial^2 \phi}{\partial x^2} = -\sin x \cosh y$, $\frac{\partial^2 \phi}{\partial y^2} = \sin x \cosh y$; $\frac{\partial^2 \phi}{\partial z^2} = -\cos(z - vt)$, $\frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = -\cos(z - vt)$

Therefore, $\frac{1}{v^2} \frac{\partial^2 \Phi}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi$

Correct answer is (a)

5. The coordinates and momenta x_i, p_i ($i = 1, 2, 3$) of a particle satisfy the canonical Poisson bracket relations $\{x_i, p_j\} = \delta_{ij}$. If $C_1 = x_2 p_3 + x_3 p_2$ and $C_2 = x_1 p_2 - x_2 p_1$ are constants of motion, and if

$C_3 = \{C_1, C_2\} = x_1 p_3 + x_3 p_1$, then

(a) $\{C_2, C_3\} = C_1$ and $\{C_3, C_1\} = C_2$

(b) $\{C_2, C_3\} = -C_1$ and $\{C_3, C_1\} = -C_2$

(c) $\{C_2, C_3\} = -C_1$ and $\{C_3, C_1\} = C_2$

(d) $\{C_2, C_3\} = C_1$ and $\{C_3, C_1\} = -C_2$

soln. $\{c_2, c_3\} = \{c_2, c_3\}_{x_1, p_1} + \{c_2, c_3\}_{x_2, p_2} + \{c_2, c_3\}_{x_3, p_3}$

$\{c_2, c_3\} = \frac{\partial c_2}{\partial x_1} \cdot \frac{\partial c_3}{\partial p_1} - \frac{\partial c_2}{\partial p_1} \cdot \frac{\partial c_3}{\partial x_1} + \dots + \dots$

$= p_2 \cdot x_3 - (-x_2) p_3 + (-p_1) \cdot 0 - x_1 \cdot 0 + 0 \cdot x_1 - 0 \cdot p_1$

$$= p_2 x_3 + x_2 p_3 = c_1$$

$$\{c_3, c_1\} = \{c_3, c_1\}_{x_1, p_1} + \{c_3, c_1\}_{x_2, p_2} + \{c_3, c_1\}_{x_3, p_3}$$

$$= \frac{\partial c_3}{\partial x_1} \cdot \frac{\partial c_1}{\partial p_1} - \frac{\partial c_3}{\partial p_1} \cdot \frac{\partial c_1}{\partial x_1} + \dots + \dots$$

$$= p_3 \cdot 0 - x_3 \cdot 0 + 0 \cdot x_3 - 0 \cdot p_2 + p_1 x_2 - x_1 \cdot p_2$$

$$= p_1 x_2 - p_2 x_1$$

$$= -c_2$$

Correct answer is (d)

66. A canonical transformation relates the old coordinates (q, p) to the new ones (Q, P) by the relations $Q = q^2$ and $P = p/2q$. The corresponding time-independent generating function is

(a) $\frac{P}{q^2}$

(b) $q^2 P$

(c) q^2/P

(d) qP^2

Soln. We have to find $F(q, P)$

We have $p = \frac{\partial F}{\partial q}$ and $Q = \frac{\partial F}{\partial P}$

From given relation $p = 2Pq$

$\therefore 2Pq = \frac{\partial F}{\partial q} \Rightarrow F = Pq^2 + c_1(P)$ (i)

From given relation $Q = q^2$

$\therefore q^2 = \frac{\partial F}{\partial P} \Rightarrow F = q^2 P + c_2(q)$ (ii)

using (i) and (ii) we get $F = q^2 P$

Correct answer is (b)

67. The time evolution of a one-dimensional dynamical system is described by

$$\frac{dx}{dt} = -(x+1)(x^2 - b^2)$$

If this has one stable and two unstable fixed points, then the parameter 'b' satisfies

(a) $0 < b < 1$

(b) $b > 1$

(c) $b < -1$

(d) $b = 2$

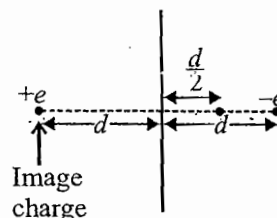
Soln. **Correct answer is (b)**

68. A charge $(-e)$ is placed in vacuum at the point $(d, 0, 0)$, where $d > 0$. The region $x \leq 0$ is filled uniformly with a metal. The electric field at the point $\left(\frac{d}{2}, 0, 0\right)$ is

(a) $-\frac{10e}{9\pi\epsilon_0 d^2}(1, 0, 0)$ (b) $\frac{10e}{9\pi\epsilon_0 d^2}(1, 0, 0)$ (c) $\frac{e}{\pi\epsilon_0 d^2}(1, 0, 0)$ (d) $-\frac{e}{\pi\epsilon_0 d^2}(1, 0, 0)$

Soln. $\vec{E} = \frac{e}{4\pi\epsilon_0 \left(\frac{d}{2}\right)^2} \hat{x} + \frac{e}{4\pi\epsilon_0 \left(\frac{3d}{2}\right)^2} \hat{x}$

$$= \frac{4e}{4\pi\epsilon_0 d^2} \hat{x} \left[1 + \frac{1}{9} \right]$$



$$\vec{E} = \frac{10e}{9\pi\epsilon_0 d^2} \hat{x}$$

Correct answer is (b)

69. An electron is in the ground state of a hydrogen atom. The probability that it is within the Bohr radius is approximately equal to

(a) 0.60 (b) 0.90 (c) 0.16 (d) 0.32

Soln. Probability that the electron will be within the Bohr radius

$$= \int_0^{a_0} \psi_0^* \psi_0 4\pi r^2 dr \quad \text{where, } \psi_0 = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

$$= \frac{1}{\pi a_0^3} 4\pi \int_0^{a_0} e^{-2r/a_0} r^2 dr = 1 - \frac{5}{e^2} = 0.32$$

Correct answer is (d)

70. A beam of light of frequency ω is reflected from a dielectric-metal interface at normal incidence. The refractive index of the dielectric medium is n and that of the metal is $n_2 = n(1+i\rho)$. If the beam is polarised parallel to the interface, then the phase change experienced by the light upon reflection is

(a) $\tan^{-1}\left(\frac{2}{\rho}\right)$ (b) $\tan^{-1}\left(\frac{1}{\rho}\right)$ (c) $\tan^{-1}\left(\frac{2}{\rho}\right)$ (d) $\tan^{-1}(2\rho)$

Soln. $E_R = \frac{n_1 - n_2}{n_1 + n_2} E_1$, $E_R = \frac{n - n(1+i\rho)}{n + n(1+i\rho)} E_1$

$$= \frac{-i\rho}{2 + i\rho} E_1 = \frac{-\rho(2i + \rho)}{4 + \rho^2} E_1$$

$$E_R = \frac{-\rho}{\sqrt{4 + \rho^2}} e^{i\phi} E_1 \quad \text{where } \phi = \tan^{-1} \frac{2}{\rho}$$

Correct answer is (c)

1. The scattering amplitude $f(\theta)$ for the potential $V(r) = \beta e^{-\mu r}$, where β and μ are positive constants, is given, in the Born approximation by

(in the following $b = 2k \sin \frac{\theta}{2}$ and $E = \frac{\hbar^2 k^2}{2m}$)

(a) $-\frac{4m\beta\mu}{\hbar^2(b^2 + \mu^2)^2}$ (b) $-\frac{4m\beta\mu}{\hbar^2 b^2(b^2 + \mu^2)}$ (c) $-\frac{4m\beta\mu}{\hbar^2 \sqrt{b^2 + \mu^2}}$ (d) $-\frac{4m\beta\mu}{\hbar^2(b^2 + \mu^2)^3}$

As high

Soln. Given : $V(r) = \beta e^{-\mu r}$ (spherically symmetric potential).

Using First Born approximation,

$$\text{Scattering amplitude } f(\theta, \phi) = -\frac{2m}{b\hbar^2} \int_0^\infty r \sin br V(r) dr$$

$$\Rightarrow f(\theta, \phi) = -\frac{2m}{b\hbar^2} \int_0^\infty r \sin br \cdot \beta e^{-\mu r} dr = -\frac{2m\beta}{b\hbar^2} \int_0^\infty e^{-\mu r} (r \sin br) dr$$

Using, Laplace transform of $r \sin br$ i.e.

$$L(r \sin br) = \int_0^\infty e^{-sr} (r \sin br) dr$$

$$\Rightarrow (-1)^1 \frac{d}{ds} \left[\frac{b}{s^2 + b^2} \right] = \int_0^\infty e^{-sr} (r \sin br) dr$$

$$\Rightarrow \frac{2sb}{(s^2 + b^2)^2} = \int_0^\infty e^{-sr} (r \sin br) dr$$

$$\text{Therefore, } f(\theta) = -\frac{4m\beta\mu}{\hbar^2 (b^2 + \mu^2)^2}$$

Correction answer is (a):

72. The ground state eigenfunction for the potential $V(x) = -\delta(x)$, where $\delta(x)$ is the delta function, is given by $\psi(x) = Ae^{-\alpha|x|}$ where A and $\alpha > 0$ are constants. If a perturbation $H' = bx^2$ is applied, the first order correction to the energy of the ground state will be

- (a) $\frac{b}{\sqrt{2}\alpha^2}$ (b) $\frac{b}{\alpha^2}$ (c) $\frac{2b}{\alpha^2}$ (d) $\frac{b}{2\alpha^2}$

Soln. First order correction to ground state energy

$$\begin{aligned} E_0^{(1)} &= \frac{\langle \psi | H' | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int_{-\infty}^{\infty} A^2 e^{-2\alpha|x|} \cdot bx^2 dx}{\int_{-\infty}^{\infty} A^2 e^{-2\alpha|x|} dx} \\ &= \frac{2 \int_0^\infty e^{-2\alpha x} bx^2 dx}{2 \int_0^\infty e^{-2\alpha x} dx} \\ &= \left[b \cdot \frac{2!}{(2\alpha)^3} \right] / \left[\frac{1}{(2\alpha)} \right] = \frac{b}{2\alpha^2} \end{aligned}$$

$$[\text{Using } \int_0^\infty e^{-ax} \cdot x^n dx = \frac{n!}{a^{n+1}}]$$

Correct answer is (d)

73. A thin infinitely long solenoid placed along the z-axis contains a magnetic flux ϕ . Which of the following vector potentials corresponds to the magnetic field at an arbitrary point (x, y, z) ?

(a) $(A_x, A_y, A_z) = \left(-\frac{\phi}{2\pi} \frac{y}{x^2 + y^2}, \frac{\phi}{2\pi} \frac{x}{x^2 + y^2}, 0 \right)$

(b) $(A_x, A_y, A_z) = \left(-\frac{\phi}{2\pi} \frac{y}{x^2 + y^2 + z^2}, \frac{\phi}{2\pi} \frac{x}{x^2 + y^2 + z^2}, 0 \right)$

(c) $(A_x, A_y, A_z) = \left(-\frac{\phi}{2\pi} \frac{x+y}{x^2 + y^2}, \frac{\phi}{2\pi} \frac{x+y}{x^2 + y^2}, 0 \right)$

(d) $(A_x, A_y, A_z) = \left(-\frac{\phi}{2\pi} \frac{x}{x^2 + y^2}, \frac{\phi}{2\pi} \frac{y}{x^2 + y^2}, 0 \right)$

Soln. We know that magnetic vector potential outside the solenoid is

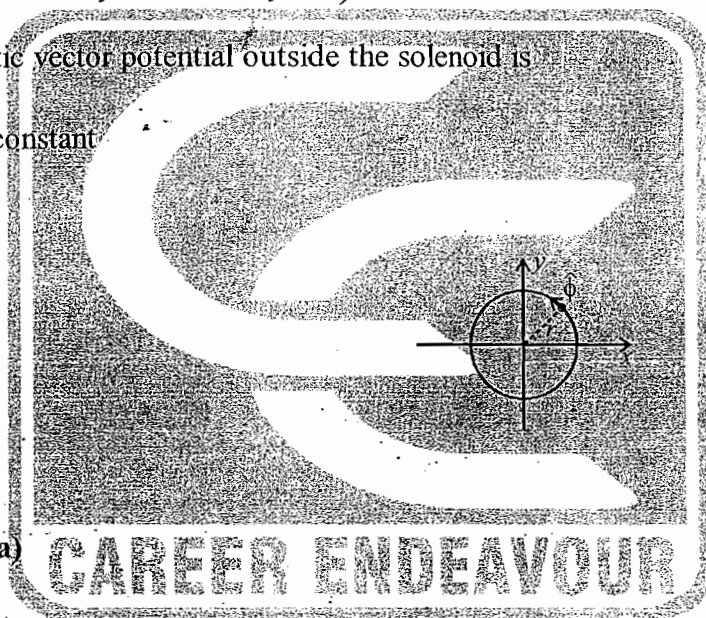
$$\vec{A} = \frac{k}{r} \hat{\phi} \quad \text{where } k \text{ is constant}$$

$$\hat{\phi} = \frac{-y\hat{i} + x\hat{j}}{r}$$

$$\therefore \vec{A} = \frac{k}{r^2} (-y\hat{i} + x\hat{j})$$

$$\hat{\phi} = \frac{k}{x^2 + y^2} (-y\hat{i} + x\hat{j})$$

Correct answer is (a)



74. The van der Waals equation of state for a gas is given by

$$\left(P + \frac{a}{V^2} \right) (V - b) = RT$$

where, P, V and T represent the pressure, volume and temperature respectively, and a and b are constant parameters. At the critical point, where all the roots of the above cubic equation are degenerate, the volume is given by

(a) $\frac{a}{9b}$ (b) $\frac{a}{27b^2}$ (c) $\frac{8a}{27bR}$ (d) $3b$

Soln. The vander wall's equation of state for a gas is

$$\left(P + \frac{a}{V^2} \right) (V - b) = RT; \quad P = \frac{RT}{V - b} - \frac{a}{V^2}$$

at critical point, $\left(\frac{\partial P}{\partial V} \right)_{V=V_c} = 0; \quad \left(\frac{\partial^2 P}{\partial V^2} \right)_{V=V_c} = 0$

$$V_c = 3b$$

Correct answer is (d)

75. An electromagnetically-shielded room is designed so that at a frequency $\omega = 10^7$ rad/s the intensity of the external radiation that penetrates the room is 1% of the incident radiation. If $\sigma = \frac{1}{2\pi} \times 10^6 (\Omega m)^{-1}$ is the conductivity of the shielding material, its minimum thickness should be (given that $\ln 10 = 2.3$)
- (a) 4.60 mm (b) 2.30 mm (c) 0.23 mm (d) 0.46 mm

Soln. Electromagnetically shielded wave equation defined as

$$E = E_0 e^{-\mu x} = E_0 e^{-x/\delta}$$

where, δ = skin depth, x = thickness

Intensity, $I \propto E^2$

$$I = KE^2$$

$$I = E^2 \quad (k=1)$$

$$I = I_0 e^{-2x/\delta}$$

Given that in question,

$$I = \frac{I_0}{100} = 1\%$$

i.e., $\frac{I_0}{100} = I_0 e^{-2x/\delta}$

$$10^{-2} = e^{-2x/\delta} \Rightarrow 10^2 = e^{2x/\delta}$$

Or, $\frac{2x}{\delta} = \ln 10^2$

$$x = \delta \ln 10 = \sqrt{\frac{2}{\mu \sigma \omega}} (\ln 2)$$

$$x = \sqrt{\frac{2}{(4\pi \times 10^{-7})}} \times \frac{1}{2\pi} \times 10^6 \times 10^7 \times (2.30) = 10^{-3} \times 230 \text{ m}$$

$$x = 2.30 \text{ mm}$$

Correct answer is (b)

